NONMATRIX CLOSED-FORM EXPRESSIONS OF THE CRAMÉR-RAO BOUNDS FOR NEAR-FIELD LOCALIZATION PARAMETERS

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ABSTRACT

Near-field source localization problem by a passive antenna array makes the assumption that the time-varying sources are located near the antenna. In this situation, the far-field assumption (planar wavefront) is no longer valid and we have to consider a more complicated model parameterized by the bearing (as in the far-field case) and by the distance, named range, between the source and a reference sensor. We can find a plethora of estimation schemes in the literature but the ultimate performance has not been fully investigated. In this paper, we derive and analyze the Cramér-Rao Bound (CRB) for a single time-varying source. In this case, we obtain nonmatrix closed-form expressions. Our approach has two advantages: (i) the computational cost for a large number of snapshots of a matrix-based CRB can be high while our approach is cheap and (ii) some useful informations can be deduced from the behavior of the bound. In particular, we show that closer is the source from the array and/or higher is the carrier frequency, better is the estimation of the range.

Index Terms– Bearing and range estimation, performance bound.

1. INTRODUCTION

Passive sources localization by an array of sensors is an important topic with a large number of applications, such as radar, seismology, digital communications, etc. Particularly, the context of far-field sources has been widely investigated in the literature and a plethora of algorithms to estimate localization parameters have been proposed [1]. In this case, the sources are assumed to be far from the array of sensors. Consequently, the propagating waves are assumed to have planar wavefront. However, when the sources are located in the so-called near-field region, the curvature of the waves impinging on the sensors can no longer be approximated. Therefore, in this scenario, each time-varying source is characterized by its bearing and its range (distance between the source and a reference sensor). We can find several estimation schemes adapted to this problem [2], but there exists a few number of works studying the optimal performance associated to this model. To characterize this performance, the Cramer-Rao Bound (CRB) is a popular mathematical tool in signal processing which expresses a lower bound on the covariance matrix of any unbiased estimator. Unlike in the far-field case [3], the CRB for the near-field localization problem has been largely understudied. One can find (see, e.g., [4], [2], [5], etc.) matrix-based expressions of the stochastic CRB (i.e., when the sources are assumed to be Gaussian) but to the best of our knowledge, no result is available concerning the CRB for unknown near-field deterministic sources. The goal of this paper is to fill this lack. Particularly, a closed-form expression of the CRB in the case of a single deterministic time-varying narrow-band source in the near-field region is given and analyzed. If we assume that we dispose of T snapshots of a single source, the number of unknown model parameter grows with T. This means that the maximum likelihood estimator will not be efficient for a large number of snapshots [3]. However, it will be efficient at high signal-to-noise ratio for a fixed number of snapshot [6]. Consequently, the computation of the associated CRB becomes time consuming and we need a nonmatrix expression of the CRB which is the cornerstone of this paper. Note that the nonmatrix CRB here proposed can be obtained only for a single source and the case of multiple-source cannot be cast into the proposed framework. The proposed CRB is given with respect to the physical parameters of the problem, i.e., bearing, range, amplitude and phase shift of the source. We provide a discussion on the CRB's behavior with respect to some parameters of the problem, i.e., carrier frequency, range, bearing and the number of sensors. Finally, simulation results are provided to validate our theoretical analysis.

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2. PROBLEM SETUP

Consider an Uniform Linear Array (ULA) of N $(N > 1)$ sensors with inter-element spacing d that receives the signal emitted by a near-field and narrow-band source. Consequently, the observation model becomes

$$x_n(t) = s(t)e^{j\tau_n} + v_n(t), \quad t = 1, \ldots, T, \; n = 0, \ldots, N-1 \quad (1)$$

where $x_n(t)$ is the observed signal at the output of the n^{th} sensor, where $s(t) = \alpha(t)e^{j(2\pi f_0 t + \psi(t))}$ [7] is the emitted signal for a carrier frequency equals to f_0 and $\alpha(t)$, $\psi(t)$ are the real amplitude and the phase shift of the source, respectively. The random process $v_n(t)$ is an additive noise and T $(T > N)$ is the number of snapshots. The time delay τ_n associated with the signal propagation time from the source to the n^{th} sensor is given by [4]

$$\tau_n = \frac{2\pi r}{\lambda}\left(\sqrt{1 + \frac{n^2 d^2}{r^2} - \frac{2nd\sin\theta}{r}} - 1\right), \quad (2)$$

where λ is the signal wavelength and r, θ are the range and the bearing of the source, respectively. It is well known that if the range is inside the so-called Fresnel region [8], i.e.,

$$0.62(d^3(N-1)^3/\lambda)^{1/2} < r < 2d^2(N-1)^2/\lambda, \quad (3)$$

then the time delay τ_n can be well approximated as follows

$$\tau_n = \omega n + \phi n^2 + O\left(\frac{d^2}{r^2}\right), \quad (4)$$

where $O(\beta)$ represents terms of order larger or equal to β, and ω and ϕ are the so-called electric angles which are connected to the physical parameters of the problem by the following relationships

$$\omega = -2\pi\frac{d}{\lambda}\sin(\theta), \quad (5)$$

$$\phi = \pi\frac{d^2}{\lambda r}\cos^2(\theta). \quad (6)$$

Then, tacking into account (4), the observation model becomes:

$$x_n(t) = s(t)e^{j\left(\omega n + \phi n^2\right)} + v_n(t). \quad (7)$$

Consequently, the observation vector can be expressed as

$$\mathbf{x}(t) = [x_1(t)\dots x_N(t)]^T = \mathbf{a}(\omega,\phi)s(t) + \mathbf{v}(t), \quad (8)$$

where $\mathbf{v}(t) = [v_1(t)\dots v_N(t)]^T$. The n^{th} element of the steering vector $\mathbf{a}(\omega,\phi)$ is given by

$$[\mathbf{a}(\omega,\phi)]_n = e^{j(\omega n + \phi n^2)}. \quad (9)$$

In the remaining of the paper, we will use the following assumptions:

- The noise is assumed to be a complex circular white Gaussian random noise with zero-mean and unknown variance σ^2
- The noise is assumed to be uncorrelated both temporally and spatially
- The unknown parameter vectors $\boldsymbol{\kappa} = [r\ \theta\ \boldsymbol{\psi}^T\ \boldsymbol{\alpha}^T\ \sigma^2]^T$ and $\boldsymbol{\xi} = [\omega\ \phi\ \boldsymbol{\psi}^T\ \boldsymbol{\alpha}^T\ \sigma^2]^T$ with $\boldsymbol{\psi} = [\psi(1)\dots\psi(T)]^T$ and $\boldsymbol{\alpha} = [\alpha(1)\dots\alpha(T)]^T$ are assumed to be deterministic.

The joint probability density function of the observation $\boldsymbol{\chi} = [\mathbf{x}^T(1)\dots\mathbf{x}^T(T)]^T$ given $\boldsymbol{\xi}$ can be written as follows :

$$p(\boldsymbol{\chi}|\,\boldsymbol{\xi}) = \frac{1}{\pi^{NT}\det(\mathbf{R})}e^{-(\boldsymbol{\chi}-\boldsymbol{\mu})^H\mathbf{R}^{-1}(\boldsymbol{\chi}-\boldsymbol{\mu})}, \quad (10)$$

where $\mathbf{R} = \sigma^2\mathbf{I}_{NT}$ and

$$\boldsymbol{\mu} = [\mathbf{a}^T(\omega,\theta)s(1)\dots\mathbf{a}^T(\omega,\theta)s(T)]^T. \quad (11)$$

The goal of the next section is to derive the CRB for the proposed model with respect to the bearing and the range.

3. CRAMÉR-RAO BOUND DEFINITION AND DERIVATION

Let $E\left\{(\hat{\boldsymbol{\xi}}-\boldsymbol{\xi})(\hat{\boldsymbol{\xi}}-\boldsymbol{\xi})^T\right\}$ be the covariance matrix of an unbiased estimate of $\boldsymbol{\xi}$, denoted by $\hat{\boldsymbol{\xi}}$ and define the Cramér-Rao Bound (CRB) [9] for the considered model. The covariance inequality principle states that under quite general/weak conditions, we have

$$\mathrm{MSE}([\hat{\boldsymbol{\xi}}]_i) = E\left\{\left([\hat{\boldsymbol{\xi}}]_i - [\boldsymbol{\xi}]_i\right)^2\right\} \geq \mathrm{CRB}([\boldsymbol{\xi}]_i), \quad (12)$$

where

$$\mathrm{CRB}([\boldsymbol{\xi}]_i) = [\mathbf{FIM}^{-1}]_{i,i}. \quad (13)$$

Then, we provide an analytical inversion of the Fisher Information Matrix (FIM) which leads to a nonmatrix closed-form expression of the CRB. Finally, by using the transformation formula, we obtain the (nonmatrix) expression of CRB according to the physical parameters (bearing and range).

3.1. Block-diagonal Fisher information matrix

Since we are working with a Gaussian observation model, the i^{th}, j^{th} element of the FIM for the parameter vector $\boldsymbol{\xi}$ can be written as [3]

$$\begin{aligned}[\mathbf{FIM}]_{i,j} = \\ & tr\left\{\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial[\boldsymbol{\xi}]_i}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial[\boldsymbol{\xi}]_j}\right\} + 2Re\left\{\frac{\partial\boldsymbol{\mu}^H}{\partial[\boldsymbol{\xi}]_i}\mathbf{R}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial[\boldsymbol{\xi}]_j}\right\} \\ & = \frac{NT}{\sigma^4}\frac{\partial\sigma^2}{\partial[\boldsymbol{\xi}]_i}\frac{\partial\sigma^2}{\partial[\boldsymbol{\xi}]_j} + \frac{2}{\sigma^2}Re\left\{\frac{\partial\boldsymbol{\mu}^H}{\partial[\boldsymbol{\xi}]_i}\frac{\partial\boldsymbol{\mu}}{\partial[\boldsymbol{\xi}]_j}\right\}, \quad (14)\end{aligned}$$

where $[\boldsymbol{\xi}]_i$, $Re\{u\}$ and $tr\{\mathbf{Z}\}$ denote the i^{th} element of $\boldsymbol{\xi}$, the real part of u and the trace of $\mathbf{Z}$, respectively. Then, the FIM for a single source is block-diagonal according to

$$\mathbf{FIM} = \left[\begin{array}{ccc|cc} f_{\omega,\omega} & f_{\omega,\phi} & \mathbf{f}_{\omega,\psi} & \mathbf{0}_{1\times T} & 0 \\ f_{\phi,\omega} & f_{\phi,\phi} & \mathbf{f}_{\phi,\psi} & \mathbf{0}_{1\times T} & 0 \\ \mathbf{f}_{\psi,\omega} & \mathbf{f}_{\psi,\phi} & \mathbf{F}_{\psi,\psi} & \mathbf{0}_{T\times T} & \mathbf{0}_{T\times 1} \\ \hline \mathbf{0}_{T\times 1} & \mathbf{0}_{T\times 1} & \mathbf{0}_{T\times T} & N\mathbf{I}_T & \mathbf{0}_{T\times 1} \\ 0 & 0 & \mathbf{0}_{1\times T} & \mathbf{0}_{1\times T} & \frac{NT}{\sigma^4} \end{array}\right], \quad (15)$$

where

$$f_{\omega,\omega} = \mathrm{SNR}\frac{N(N-1)(2N-1)}{3}, \quad (16)$$

$$f_{\phi,\phi} = \mathrm{SNR}\frac{N(N-1)(2N-1)(3N^2-3N-1)}{15}, \quad (17)$$

$$\mathbf{F}_{\psi,\psi} = \frac{2N}{\sigma^2}\mathrm{diag}(\boldsymbol{\alpha}\odot\boldsymbol{\alpha}), \quad (18)$$

$$f_{\omega,\phi} = f_{\phi,\omega} = \mathrm{SNR}\frac{N^2(N-1)^2}{2}, \quad (19)$$

$$\mathbf{f}^T_{\psi,\omega} = \mathbf{f}_{\omega,\psi} = N(N-1)(\boldsymbol{\alpha}\odot\boldsymbol{\alpha}), \quad (20)$$

$$\mathbf{f}^T_{\psi,\phi} = \mathbf{f}_{\phi,\psi} = \frac{N(N-1)(2N-1)}{3}(\boldsymbol{\alpha}\odot\boldsymbol{\alpha}), \quad (21)$$

where $\mathrm{SNR} = ||\boldsymbol{\alpha}||^2/\sigma^2$, $\odot$ stands for the Hadamard product and diag(.) is the diagonal operator. We notice that, thanks to the time-diversity of the source, $\mathbf{F}_{\alpha,\psi}$ and $\mathbf{F}_{\psi,\alpha}$ are null matrices. We also note the well-known propriety that the model signal parameters are decoupled from the noise variance. The other zero terms are due to the consideration of the real part in (14) applied to purely imaginary quantities.

3.2. Analytical inversion

Since the size of the FIM proposed in (15) depends on the number of snapshots, the numerical inversion to obtain the CRB can be a costly operation for large T. This is the reason why we provide an analytical inversion. Using an appropriate partition of the FIM and after writing analytically the expression of the inverse of the Schur complement of $\mathbf{F}_{\psi,\psi}$ [10] and after some algebraic efforts, the nonmatrix closed-form expression of the CRB for $\boldsymbol{\xi}$ associated with the model (7) can be expressed as:

$$\begin{aligned}
\mathrm{CRB}(\omega) &= \frac{6(2N-1)(8N-11)}{\mathrm{SNR}(N^2-1)N(N^2-4)}, \\
\mathrm{CRB}(\phi) &= \frac{90}{\mathrm{SNR}(N^2-1)N(N^2-4)}, \\
\mathrm{CRB}\left(\psi(t)\right) &= \frac{1}{2\alpha^2(t)\mathrm{SNR}}\frac{N^4-31N^3+48N^2-26N+2}{N^2(N+1)(N^2-4)}, \\
\mathrm{CRB}\left(\alpha(t)\right) &= \frac{\sigma^2}{2N}, \\
\mathrm{CRB}\left(\sigma^2\right) &= \frac{\sigma^4}{NT}.
\end{aligned}$$

And the cross terms are given by

$$\begin{aligned}
&\mathrm{CRB}\left(\omega,\phi\right) = \mathrm{CRB}\left(\phi,\omega\right) = -\frac{90}{\mathrm{SNR}N(N^2-4)(N+1)}, \\
&\mathbf{CRB}\left(\omega,\psi\right) = \mathbf{CRB}^T\left(\psi,\omega\right) = \frac{-9(2N-1)}{\mathrm{SNR}N(N+1)(N+2)}\boldsymbol{\gamma}^T, \\
&\mathbf{CRB}\left(\phi,\psi\right) = \mathbf{CRB}^T\left(\psi,\phi\right) = \frac{15}{N(N+1)(N+2)}\boldsymbol{\gamma}^T,
\end{aligned}$$

where $\boldsymbol{\gamma}$ is the vector of dimension $T \times 1$ filled by ones.

3.3. Vector parameter CRB for transformations

Even if the model (7) is usually used in array signal processing, its CRB relating to $\boldsymbol{\xi}$ does not bring us physical information. Then, it is interesting to analyze a CRB regarding the bearing and range. Having the $\mathbf{CRB}(\boldsymbol{\xi})$, we can easily obtain $\mathbf{CRB}(\boldsymbol{\kappa})$ by using the following formula (see [11] p. 45)

$$\mathbf{CRB}(\boldsymbol{\kappa}) = \frac{\partial \mathbf{g}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\mathbf{CRB}(\boldsymbol{\xi})\frac{\partial \mathbf{g}^T(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}, \tag{22}$$

where

$$\begin{aligned}
&\boldsymbol{\kappa} = \mathbf{g}(\boldsymbol{\xi}) = \\
&[-\arcsin(\frac{\omega\lambda}{2\pi d}) \quad \frac{\pi d^2}{\lambda\phi}\cos^2\left(\arcsin(\frac{\omega\lambda}{2\pi d})\right) \quad \boldsymbol{\psi}^T \; \boldsymbol{\alpha}^T \; \sigma^2]^T,
\end{aligned} \tag{23}$$

and where the Jacobian matrix is given by

$$\frac{\partial \mathbf{g}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} = \begin{bmatrix} \frac{\partial g_1(\boldsymbol{\xi})}{\partial \xi_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial g_2(\boldsymbol{\xi})}{\partial \xi_1} & \frac{\partial g_2(\boldsymbol{\xi})}{\partial \xi_2} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{bmatrix}, \tag{24}$$

where

$$\frac{\partial g_1(\boldsymbol{\xi})}{\partial \xi_1} = -\frac{\lambda}{2\pi d\sqrt{1-\frac{\omega^2\pi^2 d^4}{\lambda^2}}}, \tag{25}$$

$$\frac{\partial g_2(\boldsymbol{\xi})}{\partial \xi_1} = -\frac{\lambda\omega}{2\pi\phi}\cos\left(\arcsin(\frac{\lambda\omega}{2\pi d})\right)\frac{1}{\sqrt{1-\left(\frac{\lambda\omega}{2\pi d}\right)^2}}, \tag{26}$$

$$\frac{\partial g_2(\boldsymbol{\xi})}{\partial \xi_2} = -\frac{\pi d^2}{\lambda\phi^2}\cos^2\left(\arcsin(\frac{\lambda\omega}{2\pi d})\right). \tag{27}$$

Using (22) and the Jacobian above, we obtain after some tedious calculus

$$\mathrm{CRB}(\theta) = \frac{3\lambda^2}{2\mathrm{SNR}d^2\pi^2\cos^2(\theta)}\frac{(8N-11)(2N-1)}{(N^2-1)N(N^2-4)}, \tag{28}$$

$$\begin{aligned}
\mathrm{CRB}(r) &= \frac{6r^2\lambda^2}{\mathrm{SNR}\pi^2 d^4} \\
&\times\frac{15r^2+30drp_1(N)\sin(\theta)+d^2p_2(N)\sin^2(\theta)}{p_3(N)\cos^4(\theta)},
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
p_1(N) &= N-1, \\
p_2(N) &= (8N-11)(2N-1), \\
p_3(N) &= N(N^2-1)(N^2-4).
\end{aligned}$$

Note that, of course, $\mathbf{CRB}(\psi)$, $\mathbf{CRB}(\alpha)$ and $\mathrm{CRB}\left(\sigma^2\right)$ remains unchanged. And the cross terms between θ and r are as follows

$$\begin{aligned}
\mathrm{CRB}\left(\theta,r\right) &= \mathrm{CRB}\left(r,\theta\right) = \\
&-\frac{3\lambda^2 r}{\mathrm{SNR}\pi^2 d^3}\frac{15rp_1(N)+dp_2(N)\sin(\theta)}{p_3(N)\cos^3(\theta)}.
\end{aligned} \tag{30}$$

4. ANALYSIS OF THE CRB AND NUMERICAL ILLUSTRATIONS

The goal of this Section is to validate and analyze the proposed closed-form expressions given in 3.3. The context of these simulations is an ULA of 10 sensors spaced by a half-wavelength. The number of snapshots is equal to $T = 20$ and the location of the source is set as $(\theta, r) = (30^o, 6\lambda)$ (which belongs to the Fresnel region according to (3)). In Fig. 1 and Fig. 2, we compare the analytical CRB, obtained in (28) and (29), to the exact CRB, computed numerically by inverting the FIM. The variance of the noise varies between 0.1 and 1 for different values of the carrier frequency ($\lambda = \frac{c}{f_0}$, where c denote the speed of light). These figures validate our analytic expressions. Furthermore, from the closed-form expressions given in 3.3, we notice that

- The CRB are phase-invariant.
- The CRB(θ) is just bearing-dependent as in the far-field scenario w.r.t. $O(1/\cos^2(\theta))$. It means that the ULA in the near-filed case is not isotropic [12].
- The CRB(r) is bearing-dependent and range-dependent. For $\lambda, r \propto d$, the dependence w.r.t. the range is $O(r^2)$, meaning that nearer is the source better is the estimation (keeping in mind the Fresnel constraints). The dependence of the range w.r.t. the bearing is $O(1/\cos^4(\theta))$. For θ close to $\pi/2$, we observe that the CRB(r) goes to infinity but faster than the CRB(θ)(*cf.* Fig. 3).
- For a sufficient number of sensors, the CRB(θ) and the CRB(r) are $O(1/N^3)$.
- For $\lambda \propto d$, the CRB(θ) is independent of the carrier frequency f_0. This is not the case of the CRB(r). Fig. 3 shows that, for different values of bearing and for a fixed variance ($\sigma^2 = 0.5$), higher the frequency is, lower is CRB(r).

- For large N and fixed inter-spacing sensor, the CRB(θ) in the near-field case tends to the one in the far-field case given by $\frac{3\lambda^2}{\mathrm{SNR}2d^2\pi^2\cos^2(\theta)N^3}$ [3]. This is in adequation with the intuition since, due to the Fresnel constraint, large N implies large range. This corresponds to the far-field scenario.
- Note that the expression of CRB (θ, r) shows that the physical parameters of interest are strongly coupled since CRB (θ, r) is $O(1/N^3)$ as CRB(θ) and CRB(r).

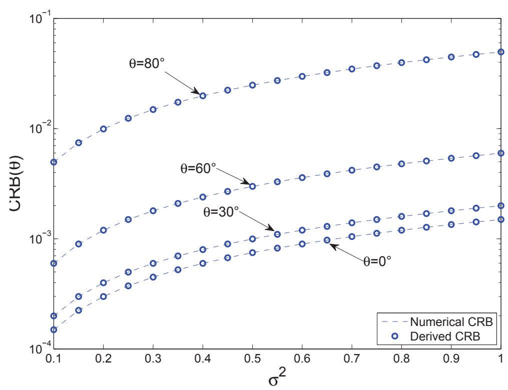

Fig. 1. The numerical and derived CRB(θ) vs. σ^2 for $(\theta, r) = (30^o, 6\lambda)$.

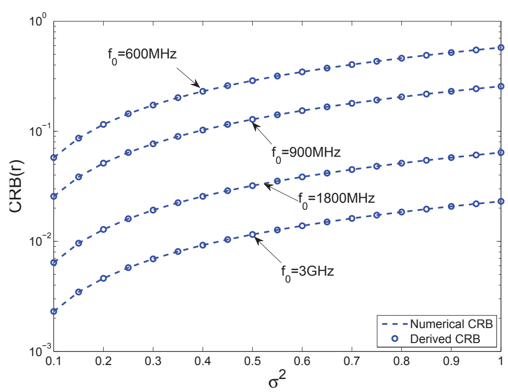

Fig. 2. The numerical and derived CRB(r) vs. σ^2 for $(\theta, r) = (30^o, 6\lambda)$ for different values of $f_0 = 600, 900, 1800, 3000$[MHz].

5. CONCLUSION

In this paper, we derived the deterministic Cramér-Rao bound in a closed-form expression for a single near-field time-varying narrowband source for the model parameters (range, bearing, amplitudes, phases). These expressions are given in nonmatrix forms which are important in order to avoid a costly FIM numerical inversion since the size of the model parameters vector increases with the number of snapshots. Moreover these expressions provide useful informations concerning the behavior of the bounds. In this way, the proposed expressions have been analyzed with respect to the physical parameters of the problem. Finally, our analytical expressions are validated by numerical simulations.

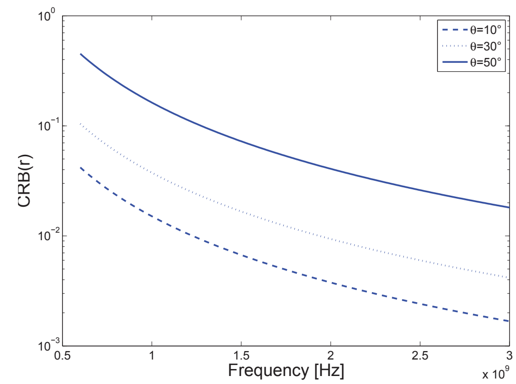

Fig. 3. CRB(r) vs. f_0 for $\sigma^2 = 0.5$ and different values of $\theta = 10^o, 30^o, 50^o$.

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