Closed-form expression of the Weiss-Weinstein bound for 3D source localization: the conditional case

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Abstract—In array processing, lower bounds are used as a benchmark to evaluate the ultimate performance of estimators. Among these bounds, the Weiss-Weinstein bound (WWB) is known as the tightest bound of Weiss-Weinstein family, and is able to predict the threshold effect of estimator's mean square error (MSE) at low signal-to-noise ratio (SNR) and/or at low number of snapshots. In this paper, we derive a closed-form expression of the WWB for 3D source localization using an arbitrary planar antenna array in the case of a deterministic known signal. The presented results are shown to be useful for system design such as array geometry optimization.

Index Terms—DOA estimation, Weiss-Weinstein bound.

I. INTRODUCTION

The passive source localization problem has been intensively investigated in the literature [1]. One of the objectives is to estimate the direction-of-arrival (DOA) of sources located in the outer space by using an array of sensors. In order to evaluate the estimation performance independently of the considered estimator, one generally uses lower bounds on the MSE [2], for example the well known Cramér-Rao bound (CRB) [3]. However, at low SNR and/or at low number of snapshots, the CRB is too optimistic. This is due to the fact that the CRB does not take into account the parameter support and that estimators are generally biased in such non-asymptotic area. For these reasons, we are here interested to derive a bound which is more relevant, the so-called Weiss-Weinstein bound (WWB) [4]. The WWB is a Bayesian bound known to be one of the tightest bound of the Weiss-Weinstein family [5], [6] and which can predict the MSE of estimators over all the SNR range.

Let us note that, in array processing, the source signals are modeled as either random Gaussian process or deterministic quantities, which are referred to the unconditional or conditional observation models, respectively. Particularly, under the conditional assumption, the signal waveforms can be assumed either unknown or known. While the conditional observation model with unknown waveform seems more challenging, the conditional model with known waveforms signals can be found in several applications such as in mobile telecommunication. Indeed, the information concerning the waveforms signal helps to improve the estimation accuracy and also to simplify the implementation (see *e.g.* [7], [8], [9], [10], and [11]).

Concerning the applications of the WWB in array processing, surprisingly, to the best of our knowledge, almost all the previous works are related to the unconditional assumption. In [12], for the first time, the WWB has been evaluated by way of simulations and has been compared to the MSE of the MUSIC algorithm and classical Beamforming using an 8×8 element array antenna. In [13], the authors have introduced a numerical comparison between the Bayesian CRB, the Ziv-Zakai bound (ZZB) and the WWB for DOA estimation. In [14], numerical simulations of the WWB to optimize sensor positions for non-uniform linear arrays have been presented.

In [15], by considering the matched-field estimation problem, the authors have derived a semi closed-form expression of the WWB for the DOA estimation. Indeed, the integration over the *prior* probability density function was not performed. To the best of our knowledge, the context of conditional observation model (with known waveforms) is available only in [16], where a closed-form expression of the WWB is given in the simple case of spectral analysis. One can also note that the ZZB, which is also a tight bound has been derived in [17], in the context of DOA estimation and under the unconditional observation model. Of course, if we are only interested by the SNR threshold prediction, deterministic bounds such as the Barankin Bound have also been investigated [18], [19].

In this paper, we derive a closed-form expression of the WWB under the conditional observation model for the 3D source localization problem using an arbitrary antenna array. As a by-product, the closed-form expression of the WWB for the 2D source localization problem using a linear antenna array (not necessarily uniform) is also given. Note that in our model, since the angle-of-arrivals are assumed to have an uniform *prior* probability density function, the classical Bayesian CRB does not exist. Consequently, ours results are also useful to analyze the asymptotic area. Finally, the proposed bound is used in the context of array geometry design of a V-shaped array [20].

The paper is organized as follows. In Sec. II, the observation model is presented. In Sec. III, we derive the closed-form expression of the WWB for the aforementioned model and in the particular case of 2D source localization. In Sec. IV, some numerical simulations are presented, and we analyze the V-shaped array design. Finally, the conclusions are given in Sec. V.

II. MODEL SETUP

We consider the localization of a single narrow-band source using an arbitrary planar array. This array consists of M identical and omnidirectional sensors. The source is assumed to be in the far-field area. The parameters of interest are the elevation and azimuth angles, denoted θ and ϕ, respectively (see Fig. 1). For mathematical convenience, we will consider the estimation of $u = \sin\theta\cos\phi$ and $v = \sin\theta\sin\phi$. Then, the parameter vector becomes $\boldsymbol{\Theta} = [u \; v]^T$. It will be assumed that both u and v have a *prior* uniform distribution over $[-1, 1]$. Moreover, u and v are assumed to be statistically independent *i.e.* $p(\boldsymbol{\Theta}) = p(u)p(v)$. The known positions of each sensor (*w.r.t.* the Cartesian coordinates) are collected in the following matrix $\mathbf{D} = [\mathbf{d}_1 \ldots \mathbf{d}_M]$, where $\mathbf{d}_i = [d_{xi} \; d_{yi}]^T$ with $i = 1 \ldots M$.

Let $\mathbf{y}(t)$, $s(t)$ and $\mathbf{n}(t)$ denote the output signal vector, the source signal and the noise vector at the tth observation, respectively, with $t = 1 \ldots T$, and where T denotes the number of snapshots. The observation model is then given by

$$\mathbf{y}(t) = [y_1(t) \ldots y_M(t)]^T = \mathbf{a}(\boldsymbol{\Theta})s(t) + \mathbf{n}(t), \qquad (1)$$

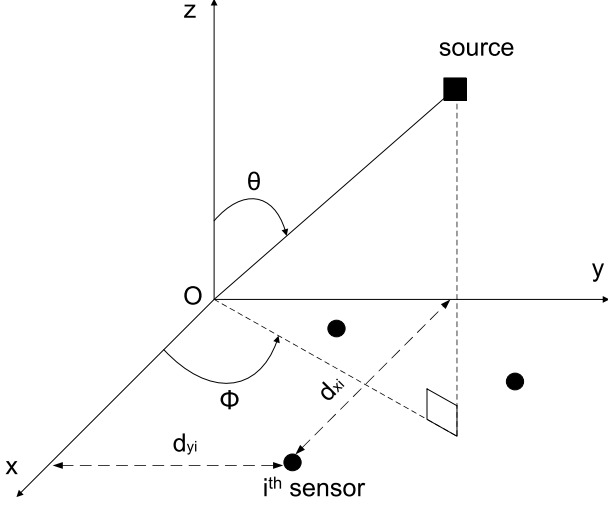where $\mathbf{a}(\boldsymbol{\Theta})$ denotes the $M \times 1$ array steering vector with the ith

Fig. 1. 3D source localization using a planar array.

element given by

$$\{\mathbf{a}(\boldsymbol{\Theta})\}_i = \exp\left(j\frac{2\pi}{\lambda}\mathbf{d}_i^T\boldsymbol{\Theta}\right)$$
$$= \exp\left(j\frac{2\pi}{\lambda}(d_{xi}u + d_{yi}v)\right). \quad (2)$$

The noise vector is assumed to be Gaussian, circular, independent and identically distributed, with zero mean and covariance matrix $\sigma_n^2\mathbf{I}$. The source signal $s(t)$ is assumed to be deterministic and known. These assumptions are justified in [7]–[11].

According to the previous assumptions, it is clear that the observations are Gaussian with parameterized mean $\mathbf{a}(\boldsymbol{\Theta})s(t)$ and covariance matrix $\sigma_n^2\mathbf{I}$. Therefore, the likelihood function of the observations $\mathbf{Y} = [\mathbf{y}(1)\dots\mathbf{y}(T)]$ is given by

$$p(\mathbf{Y};\boldsymbol{\Theta}) = \frac{1}{(\pi\sigma_n^2)^{MT}}\exp\left(-\frac{1}{\sigma_n^2}\sum_{t=1}^{T}||\mathbf{y}(t) - \mathbf{a}(\boldsymbol{\Theta})s(t)||^2\right). \quad (3)$$

In the following section, a closed-form expression of the WWB will be proposed under the aforementioned assumptions.

III. WWB FOR 3D AND 2D SOURCE LOCALIZATION

Note that we will work with a simplified expression of the WWB. Indeed, the general form of WWB requires an optimization over some parameters, called "test-points" and another parameter denoted s. As proposed in [21], we will let $s = 1/2$ and we will use one test-point per parameter. Such assumptions have been observed in [2], [15], and [16].

A. General WWB derivation

The expression of the WWB is given by [21]

$$\mathbf{WWB} = \sup_{h_u, h_v}\mathbf{H}\mathbf{G}^{-1}\mathbf{H}^T, \quad (4)$$

where $\mathbf{H}$ denotes the test-point matrix

$$\mathbf{H} = [\mathbf{h}_u \ \mathbf{h}_v] = \begin{bmatrix} h_u & 0 \\ 0 & h_v \end{bmatrix}. \quad (5)$$

Therefore, we have

$$\boldsymbol{\Theta} + \mathbf{h}_u = \begin{bmatrix} u + h_u \\ v \end{bmatrix}, \quad (6)$$

and

$$\boldsymbol{\Theta} + \mathbf{h}_v = \begin{bmatrix} u \\ v + h_v \end{bmatrix}. \quad (7)$$

The elements of the 2×2 matrix $\mathbf{G}$ are given by

$$\{\mathbf{G}\}_{kl} = \frac{2\left(\eta\left(\mathbf{h}_k, \mathbf{h}_l\right) - \eta\left(\mathbf{h}_k, -\mathbf{h}_l\right)\right)}{\eta\left(\mathbf{h}_k, \mathbf{0}\right)\eta\left(\mathbf{0}, \mathbf{h}_l\right)}, \quad (8)$$

with $\{k, l\} \in \{u, v\}^2$, where we define

$$\eta\left(\boldsymbol{\alpha}, \boldsymbol{\beta}\right) = \int_{\Gamma}\int_{\Omega}\sqrt{p\left(\mathbf{Y}, \boldsymbol{\Theta} + \boldsymbol{\alpha}\right)p\left(\mathbf{Y}, \boldsymbol{\Theta} + \boldsymbol{\beta}\right)}d\mathbf{Y}d\boldsymbol{\Theta}, \quad (9)$$

and where Ω and Γ denote the observation space and the parameter space, respectively. Since $p(\mathbf{Y}, \boldsymbol{\Theta}) = p(\mathbf{Y};\boldsymbol{\Theta})p(\boldsymbol{\Theta})$, the function $\eta\left(\boldsymbol{\alpha}, \boldsymbol{\beta}\right)$ can be rewritten as

$$\eta\left(\boldsymbol{\alpha}, \boldsymbol{\beta}\right) = \int_{\Gamma}\kappa\left(\boldsymbol{\alpha}, \boldsymbol{\beta};\boldsymbol{\Theta}\right)\sqrt{p\left(\boldsymbol{\Theta} + \boldsymbol{\alpha}\right)p\left(\boldsymbol{\Theta} + \boldsymbol{\beta}\right)}d\boldsymbol{\Theta}, \quad (10)$$

where we define

$$\kappa\left(\boldsymbol{\alpha}, \boldsymbol{\beta};\boldsymbol{\Theta}\right) = \int_{\Omega}\sqrt{p\left(\mathbf{Y};\boldsymbol{\Theta} + \boldsymbol{\alpha}\right)p\left(\mathbf{Y};\boldsymbol{\Theta} + \boldsymbol{\beta}\right)}d\mathbf{Y}. \quad (11)$$

From (3), we have

$$\kappa\left(\boldsymbol{\alpha}, \boldsymbol{\beta};\boldsymbol{\Theta}\right) = \frac{1}{(\pi\sigma_n^2)^{MT}}\int_{\Omega}\exp\left(-\frac{1}{2\sigma_n^2}\sum_{t=1}^{T}\zeta\left(\boldsymbol{\alpha}, \boldsymbol{\beta}\right)\right)d\mathbf{Y}, \quad (12)$$

where we set

$$\zeta\left(\boldsymbol{\alpha}, \boldsymbol{\beta}\right) = (\mathbf{y}(t) - \mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\alpha})s(t))^H\left(\mathbf{y}(t) - \mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\alpha})s(t)\right)$$
$$+ (\mathbf{y}(t) - \mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\beta})s(t))^H\left(\mathbf{y}(t) - \mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\beta})s(t)\right). \quad (13)$$

Let us set

$$\mathbf{x}(t) = \mathbf{y}(t) - \frac{1}{2}\left(\mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\alpha})s(t) + \mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\beta})s(t)\right), \quad (14)$$

then

$$\mathbf{x}^H(t)\mathbf{x}(t) = \frac{\zeta\left(\boldsymbol{\alpha}, \boldsymbol{\beta}\right)}{2} - \psi(t), \quad (15)$$

where

$$\psi(t) = $$
$$\frac{||s(t)||^2}{4}\left(\mathbf{a}^H(\boldsymbol{\Theta} + \boldsymbol{\beta})\mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\alpha}) + \mathbf{a}^H(\boldsymbol{\Theta} + \boldsymbol{\alpha})\mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\beta})\right.$$
$$\left.- \mathbf{a}^H(\boldsymbol{\Theta} + \boldsymbol{\alpha})\mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\alpha}) - \mathbf{a}^H(\boldsymbol{\Theta} + \boldsymbol{\beta})\mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\beta})\right). \quad (16)$$

Consequently, we have

$$\kappa\left(\boldsymbol{\alpha}, \boldsymbol{\beta};\boldsymbol{\Theta}\right) = \frac{1}{(\pi\sigma_n^2)^{MT}}\int_{\Omega}\exp\left(-\sum_{t=1}^{T}\frac{\mathbf{x}^H(t)\mathbf{x}(t) + \psi(t)}{\sigma_n^2}\right)d\mathbf{X}$$
$$= \exp\left(-\sum_{t=1}^{T}\frac{\psi(t)}{\sigma_n^2}\right), \quad (17)$$

since

$$\frac{1}{(\pi\sigma_n^2)^{MT}}\int_{\Omega}\exp\left(-\sum_{t=1}^{T}\frac{\mathbf{x}^H(t)\mathbf{x}(t)}{\sigma_n^2}\right)d\mathbf{X} = 1. \quad (18)$$

From (2), we have

$$\mathbf{a}^H(\boldsymbol{\Theta} + \boldsymbol{\beta})\mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\alpha}) = \sum_{i=1}^{M}\exp\left(j\frac{2\pi}{\lambda}\mathbf{d}_i^T(\boldsymbol{\beta} - \boldsymbol{\alpha})\right), \quad (19)$$

$$\mathbf{a}^H(\boldsymbol{\Theta} + \boldsymbol{\alpha})\mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\beta}) = \sum_{i=1}^{M}\exp\left(j\frac{2\pi}{\lambda}\mathbf{d}_i^T(\boldsymbol{\alpha} - \boldsymbol{\beta})\right), \quad (20)$$

and

$$\mathbf{a}^H(\boldsymbol{\Theta} + \boldsymbol{\alpha})\mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\alpha}) = \mathbf{a}^H(\boldsymbol{\Theta} + \boldsymbol{\beta})\mathbf{a}(\boldsymbol{\Theta} + \boldsymbol{\beta}) = M. \quad (21)$$

Consequently, a closed-form expression of $\kappa(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{\Theta})$ is given by

$$\kappa(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{\Theta}) = \exp\left(\frac{1}{2\sigma_n^2} \sum_{t=1}^{T} ||s(t)||^2 \left(M - \sum_{i=1}^{M} \cos\left(j\frac{2\pi \mathbf{d}_i^T(\boldsymbol{\beta}-\boldsymbol{\alpha})}{\lambda}\right)\right)\right). \quad (22)$$

Since (22) does not depend on $\boldsymbol{\Theta}$, (10) can be rewritten as

$$\eta(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \kappa(\boldsymbol{\alpha}, \boldsymbol{\beta}; \boldsymbol{\Theta}) \int_\Gamma \sqrt{p(\boldsymbol{\Theta} + \boldsymbol{\alpha}) p(\boldsymbol{\Theta} + \boldsymbol{\beta})} d\boldsymbol{\Theta}. \quad (23)$$

The integral in (23) can be easily calculated by noticing that both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can take values from $\{\pm\mathbf{h}_u; \pm\mathbf{h}_v; \mathbf{0}\}$ and that u and v have both a uniform prior, leading to

$$\int_\Gamma \sqrt{p(\boldsymbol{\Theta} + \boldsymbol{h}_u) p(\boldsymbol{\Theta} + \boldsymbol{h}_v)} d\boldsymbol{\Theta} = \frac{1}{2}(2 - |h_u|)(2 - |h_v|), \quad (24)$$

$$\int_\Gamma \sqrt{p(\boldsymbol{\Theta} + \boldsymbol{h}_i) p(\boldsymbol{\Theta} + \boldsymbol{h}_i)} d\boldsymbol{\Theta} = \frac{1}{2}(2 - |h_i|), \quad (25)$$

$$\int_\Gamma \sqrt{p(\boldsymbol{\Theta} + \boldsymbol{h}_i) p(\boldsymbol{\Theta} - \boldsymbol{h}_i)} d\boldsymbol{\Theta} = \frac{1}{2}(2 - 2|h_i|), \quad (26)$$

and

$$\int_\Gamma \sqrt{p(\boldsymbol{\Theta} + \boldsymbol{h}_i) p(\boldsymbol{\Theta})} d\boldsymbol{\Theta} = \frac{1}{2}(2 - |h_i|), \quad (27)$$

where $i = \{u, v\}$.

B. 3D source localization

Let us set $C_{SNR} = \frac{1}{\sigma_n^2} \sum_{t=1}^{T} ||s(t)||^2$. By applying (22)-(27) into (8), the analytic expressions of the elements of matrix $\mathbf{G}$ are given by (28)-(30) shown on the top of the next page. Due to the lack of space, the inversion of the 2×2 matrix $\mathbf{G}$ and the multiplication by the matrix of test-points is not reported here.

C. 2D source localization

In the context of the 2D source localization using a linear (possibly non-uniform) antenna array, the parameter of interest is only the elevation angle θ (*i.e.* $\phi = 0$). For mathematical convenience, we consider the estimation of $u = \sin\theta$. Again we assume that u follows a uniform distribution over $[-1; 1]$. Without loss of generality, we suppose that the linear antenna lays on the Ox axis. The ith element of the steering vector in this case is then given by

$$\{\mathbf{a}(\boldsymbol{\Theta})\}_i = \exp\left(j\frac{2\pi}{\lambda} d_{xi} u\right). \quad (31)$$

Moreover, we only use one test-point h_u

By applying (22), (23) and (25)-(27) into (8), a closed-form expression of the WWB is given by (32), shown on the top of the next page. The WWB is finally obtained by maximization over the test-point.

IV. SIMULATION RESULTS

First, we consider the 2D source localization using an uniform linear array consisting of 10 sensors. The inter-sensors space is a half-wavelength, and the number of snapshots is $T = 20$. Fig. (2) shows the MSE of the MAP estimator versus the WWB. The MSE of the MAP is obtained over 10000 Monte Carlo trials, and the WWB is obtained by numerical maximization over the test-point. One can observe that the WWB predicts the threshold effect and that the WWB coincides with the MSE of the MAP in the asymptotic region. Let us remind that, for our model, the BCRB does not exist and that, consequently, the proposed WWB is also useful to characterize the asymptotic regime.
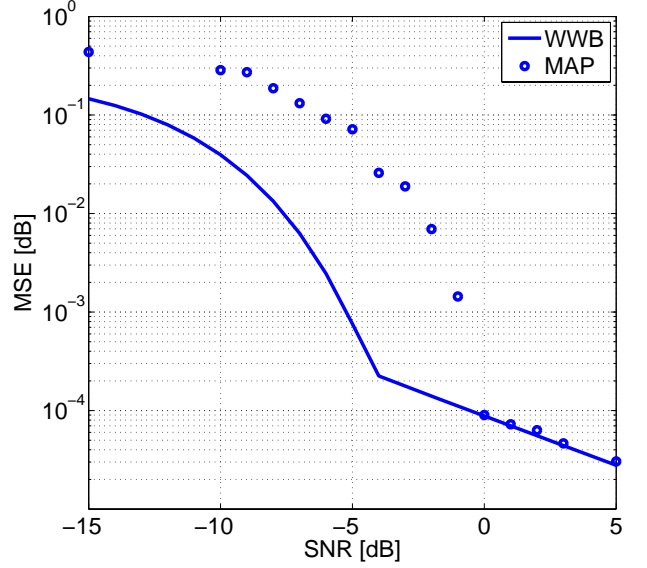

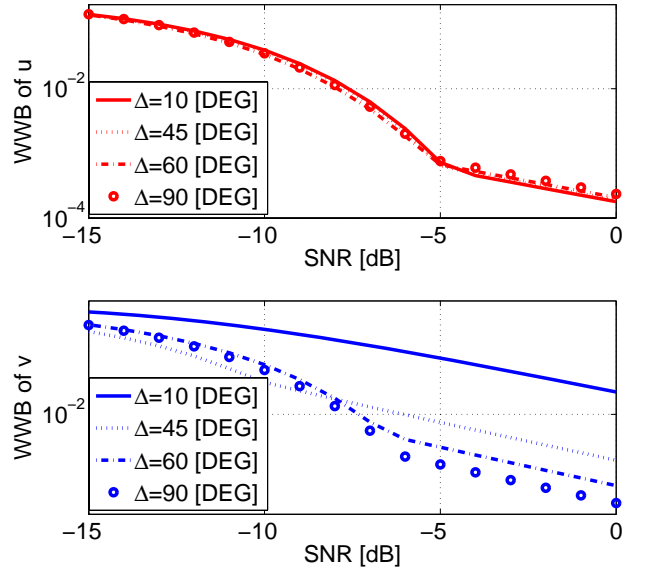Fig. 2. The performance of the MAP versus the WWB.

Fig. 3. WWB of V-shaped array w.r.t. the opening angle Δ.

Second, the WWB is used as a tool in order to study the impact of the antenna geometry on the performance estimation. In this case, we will consider 3D source localization using the V-shaped antenna [20]. Indeed, it has been shown that this kind of array is often able to outperform other classical planar arrays, more particularly the uniform circular array [22]. This array is made from two branches of ULA arrays and we denote Δ the angle between these two branches. The V-shaped antenna consists of two uniform linear arrays with 6 sensors located on each branches and one sensor located at the origin. The sensors are equalspaced by a half-wavelength. The number of snapshots is $T = 20$. Fig. (3) shows the behavior of the WWB w.r.t. the opening angle Δ. One can observe that when Δ varies, the estimation performance concerning the estimation of parameter u varies slightly. On the contrary, the estimation performance con-

$$\{\mathbf{G}\}_{uu} = \frac{2\left(1 - \frac{|h_u|}{2}\right) - 2(1 - |h_u|)\exp\left(-\frac{C_{SNR}}{2}\left(M - \sum_{k=1}^{M}\cos\left(\frac{4\pi}{\lambda}d_{xk}h_u\right)\right)\right)}{\left(1 - \frac{|h_u|}{2}\right)^2\exp\left(-C_{SNR}\left(M - \sum_{k=1}^{M}\cos\left(\frac{2\pi}{\lambda}d_{xk}h_u\right)\right)\right)}, \tag{28}$$

$$\{\mathbf{G}\}_{vv} = \frac{2\left(1 - \frac{|h_v|}{2}\right) - 2(1 - |h_v|)\exp\left(-\frac{C_{SNR}}{2}\left(M - \sum_{k=1}^{M}\cos\left(\frac{4\pi}{\lambda}d_{yk}h_v\right)\right)\right)}{\left(1 - \frac{|h_v|}{2}\right)^2\exp\left(-C_{SNR}\left(M - \sum_{k=1}^{M}\cos\left(\frac{2\pi}{\lambda}d_{yk}h_v\right)\right)\right)}, \tag{29}$$

$$\{\mathbf{G}\}_{vu} = \frac{\left(2\exp\left(-\frac{C_{SNR}}{2}\left(M - \sum_{k=1}^{M}\cos\left(\frac{2\pi}{\lambda}(d_{xk}h_u - d_{yk}h_v)\right)\right)\right) - 2\exp\left(-\frac{C_{SNR}}{2}\left(M - \sum_{k=1}^{M}\cos\left(\frac{2\pi}{\lambda}(d_{xk}h_u + d_{yk}h_v)\right)\right)\right)\right)}{\exp\left(-\frac{C_{SNR}}{2}\left(2M - \sum_{k=1}^{M}\cos\left(\frac{2\pi}{\lambda}d_{xk}h_u\right) - \sum_{k=1}^{M}\cos\left(\frac{2\pi}{\lambda}d_{yk}h_v\right)\right)\right)}. \tag{30}$$

$$WWB = \sup_{h_u} \frac{h_u^2\left(1 - \frac{|h_u|}{2}\right)^2\exp\left(-C_{SNR}\left(M - \sum_{k=1}^{M}\cos\left(\frac{2\pi}{\lambda}d_{xk}h_u\right)\right)\right)}{2\left(1 - \frac{|h_u|}{2}\right) - 2\left(1 - |h_u|\right)\exp\left(-\frac{1}{2}C_{SNR}\left(M - \sum_{k=1}^{M}\cos\left(\frac{4\pi}{\lambda}d_{xk}h_u\right)\right)\right)}. \tag{32}$$

cerning the estimation of parameter v is strongly dependent on Δ. When Δ increases from $0°$ to $90°$, the WWB of v decreases, as well as the SNR threshold. Fig. (3) also shows that $\Delta = 90°$ is the optimal value, which is different with the optimal value $\Delta = 53.13°$ in [22] since the assumptions concerning the source signal are not the same.

V. CONCLUSION

In this paper, we have derived a closed-form expression of WWB for the 3D source localization under the conditional observation model. The presented results provide a useful tool to approximate the estimator performance behavior and to predict the threshold effect. For example, these bounds have been used to find the optimal angle of the so-called V-shaped array.

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