

**STATISTICAL RESOLUTION LIMIT:
 APPLICATION TO PASSIVE POLARIZED SOURCE LOCALIZATION**

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ABSTRACT

This paper considers the evaluation of the so-called Co-centered Orthogonal Loop and Dipole Uniform and Linear Array (COLD-ULA) performance by mean of the Statistical Resolution Limit (SRL). The SRL addressed herein is based on the estimation accuracy. Toward this end, non-matrix closed form expressions of the Cramér-Rao Bound (CRB) are derived and thus, the SRL is deduced by an adequate change of variable formula. Finally, concluding remarks and a comparison between the SRL of the COLD-ULA and the ULA are given. In particular, we show that, in the case where the sources are orthogonal, the SRL for the COLD-ULA is equal to the SRL for the ULA, meaning that it is not a function of polarisation parameters. Furthermore, thanks to the derived SRL, we show that generally the performance of the COLD-ULA is better than the performance of the ULA.

1. INTRODUCTION

Passive polarized source localization by an array of sensors is an important topic in a large number of applications especially in wireless communication [1] and seismology [2]. In this context, one can find several estimation schemes. For example, in [2], [3], [4] and [5] the authors proposed an algorithm based on the shift-invariance property, the Maximum Likelihood Estimator (MLE), the ESPRIT algorithm and the MODE algorithm for polarized far-field narrow-band source localization, respectively.

However, the optimal performance, associated to this model, has not been fully investigated. In particular, the SRL on the signal parameters is an essential tool in the evaluation of system performance [6–10]. To the best of our knowledge, no results are available concerning the SRL for such a model.

The goal of this paper is to fill this lack. More precisely, the challenge herein is to determine the minimum Direction Of Arrivals (DOA) separation between two polarized sources that allows a correct sources resolvability

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for a specific array of sensors, adequate to the localization of polarized sources, called the COLD-ULA [5]. There exists essentially three approaches to determine the SRL: (1) based on the estimation accuracy [7], (2) based on the detection theory [9] and (3) based on the study of the spectral function for each estimation method [11]. In this paper we consider the SRL based on the estimation accuracy. The CRB does not directly point out the best resolution that can be achieved by an unbiased estimator. However, it expresses a lower bound on the covariance matrix of any unbiased estimator, thus it can be used to obtain the SRL. Smith defined the SRL, for pole estimation problem, as the pole separation that is greater than its standard deviation estimation [7]. In this paper, the Smith criterion will be applied to the deterministic polarized source localization. In this case, the CRB will be efficient (in the sense of the deterministic MLE) at high Signal to Noise Ratio (SNR) for a fixed number of snapshot [12]. Consequently, the CRBs and the SRL expressions derived herein, are valid under these conditions.

In the following, the CRB for the considered model is derived in nonmatrix closed form expressions [13], taking advantage of these expressions, the SRL is deduced for the COLD-ULA and compared with the SRL of the ULA. Finally, concluding remarks and comparisons between the SRL of the COLD-ULA and the ULA are given.

2. MODEL SETUP

Consider a COLD-ULA of L COLD sensors (a COLD sensor is formed by a loop and a dipole) with interelement spacing d that receives a signal emitted by M radiating far-field and narrowband sources. Assuming that the array and the incident signals are coplanar [5], i.e., the elevation is fixed to $\frac{\pi}{2}$, the signal model observed on the ℓ -th COLD sensor at the t -th snapshot is given by [5, 14]

$$\mathbf{x}_\ell(t) = \begin{bmatrix} \hat{x}_\ell(t) \\ \check{x}_\ell(t) \end{bmatrix} = \sum_{m=1}^M \alpha_m(t) \mathbf{u}_m z_m^\ell + \mathbf{v}_\ell(t),$$

$$t \in [1 \dots N], \quad \ell \in [0 \dots L - 1]$$

where N is the number of snapshots, $z_m = e^{i\frac{2\pi}{\lambda}d\sin(\theta_m)}$ denotes the spatial phase factor in which θ_m and λ are the azimuth of the m -th source and the wavelength, respectively. The time-varying source is given by¹ $\alpha_m(t) = a_m e^{i(2\pi f_0 t + \phi_m(t))}$ in which a_m is the non-zero real amplitude, $\phi_m(t)$ is the time-varying modulating phase and f_0 denotes the carrier frequency of the incident wave. The additive thermal noise is denoted by

$$\mathbf{v}_\ell(t) = [\hat{v}_\ell(t) \quad \check{v}_\ell(t)]^T$$

in which $\hat{v}_\ell(t)$ and $\check{v}_\ell(t)$ are random process. The polarization state vector \mathbf{u}_m is given by

$$\mathbf{u}_m = \begin{bmatrix} \frac{2i\pi A_{sl}}{\lambda} \cos(\rho_m) \\ -L_{sd} \sin(\rho_m) e^{i\psi_m} \end{bmatrix}$$

where $\rho_m \in [0, \pi/2]$ and $\psi_m \in [-\pi, \pi]$ are the polarization state parameters. In a modeling point of view, each dipole in the array is assumed to be a short dipole with the same length L_{sd} and each loop is assumed to be a short loop with the same area A_{sl} . Under this assumptions, the total output vector received by the COLD-ULA at the t -th snapshot can be written as follows

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{x}_0(t) \\ \vdots \\ \mathbf{x}_{L-1}(t) \end{bmatrix} = \sum_{m=1}^M \mathbf{A}_m(t) \mathbf{d}_m + \begin{bmatrix} \mathbf{v}_0(t) \\ \vdots \\ \mathbf{v}_{L-1}(t) \end{bmatrix} \quad (1)$$

where $\mathbf{A}_m(t) = \mathbf{I}_L \otimes (\alpha_m(t) \mathbf{u}_m)$ is of size $(2L) \times L$ in which the operator \otimes stands for the Kronecker product and the steering vector is defined by

$$\mathbf{d}_m = [1 \quad e^{i\frac{2\pi}{\lambda}d\sin(\theta_m)} \quad \dots \quad e^{i(L-1)\frac{2\pi}{\lambda}d\sin(\theta_m)}]^T.$$

Since the problem addressed herein is to derive the SRL based on the CRB for the proposed model, we first start by deriving the CRB for (1) in the case of two known sources.

3. DETERMINISTIC CRAMÉR-RAO BOUND DERIVATION

In the remaining of the paper, we will use the following assumptions:

- A1.** The noise is assumed to be a complex circular white Gaussian random noise with zero-mean and unknown variance σ^2 ,
- A2.** The noise is assumed to be uncorrelated both temporally and spatially,
- A3.** The sources are assumed to be deterministic where the unknown parameters vector is $\boldsymbol{\xi} = [\omega_1 \quad \omega_2 \quad \sigma^2]^T$ in which $\omega_i = \frac{2\pi}{\lambda}d\sin(\theta_i)$,
- A4.** Furthermore, in a modeling point of view, we can assume, without loss of generality, that

$$L_{sd} = \frac{2\pi A_{sl}}{\lambda} = 1.$$

¹Note that this source model is commonly used in many digital communication systems (see [1, 5] and the references therein).

Using **A1.** and **A2.** the joint probability density function of the observation $\boldsymbol{\chi} = [\mathbf{y}^T(1) \quad \dots \quad \mathbf{y}^T(N)]^T$ given $\boldsymbol{\xi}$ can be written as follows

$$p(\boldsymbol{\chi} | \boldsymbol{\xi}) = \frac{1}{\pi^{2NL} \det(\mathbf{R})} e^{-(\boldsymbol{\chi} - \boldsymbol{\mu})^H \mathbf{R}^{-1} (\boldsymbol{\chi} - \boldsymbol{\mu})},$$

where $\mathbf{R} = \sigma^2 \mathbf{I}_{2NL}$ and

$$\boldsymbol{\mu} = \sum_{m=1}^M [\mathbf{A}_m^T(1) \quad \dots \quad \mathbf{A}_m^T(L)]^T \otimes \mathbf{d}_m.$$

Let $E \{ (\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})^T \}$ be the covariance matrix of an unbiased estimate of $\boldsymbol{\xi}$, denoted by $\hat{\boldsymbol{\xi}}$ and define the CRB for the considered model. The covariance inequality principle states that under quite general/weak conditions $\text{MSE}([\hat{\boldsymbol{\xi}}]_i) = E \left\{ \left([\hat{\boldsymbol{\xi}}]_i - [\boldsymbol{\xi}]_i \right)^2 \right\} \geq \text{CRB}([\boldsymbol{\xi}]_i)$, where $\text{CRB}([\boldsymbol{\xi}]_i) = [\mathbf{FIM}^{-1}(\boldsymbol{\xi})]_{i,i}$ in which $\mathbf{FIM}(\boldsymbol{\xi})$ denotes the Fisher Information Matrix regarding to the vector parameter $\boldsymbol{\xi}$.

Since we are working with a Gaussian observation model (assumption **A1.**), the $i^{\text{th}}, j^{\text{th}}$ element of the FIM for the parameter vector $\boldsymbol{\xi}$ can be written as [11]

$$[\mathbf{FIM}(\boldsymbol{\xi})]_{i,j} = \frac{NL}{\sigma^4} \frac{\partial \sigma^2}{\partial [\boldsymbol{\xi}]_i} \frac{\partial \sigma^2}{\partial [\boldsymbol{\xi}]_j} + \frac{2}{\sigma^2} \Re \left\{ \frac{\partial \boldsymbol{\mu}^H}{\partial [\boldsymbol{\xi}]_i} \frac{\partial \boldsymbol{\mu}}{\partial [\boldsymbol{\xi}]_j} \right\}$$

where $[z]_i$ and $\Re \{z\}$ denote the i^{th} element of z and the real part of z , respectively. Then, the FIM for the proposed model is block-diagonal according to

$$\mathbf{FIM}(\boldsymbol{\xi}) = \frac{2}{\sigma^2} \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \frac{NL}{2\sigma^2} \end{bmatrix} \quad (2)$$

where

$$[\mathbf{F}]_{mp} = \Re \left\{ \frac{\partial \boldsymbol{\mu}^H}{\partial \omega_m} \frac{\partial \boldsymbol{\mu}}{\partial \omega_p} \right\} = N \Re \{ r_N (\mathbf{u}_m^H \mathbf{u}_p \mathbf{d}_m^H \mathbf{D}^2 \mathbf{d}_p + k) \} \quad (3)$$

in which $\mathbf{D} = \text{diag}\{0, \dots, L-1\}$,

$$k = \frac{\partial (\mathbf{u}_m)^H}{\partial \omega_m} \frac{\partial \mathbf{u}_p}{\partial \omega_p} \mathbf{d}_m^H \mathbf{d}_p - i \mathbf{u}_m^H \frac{\partial \mathbf{u}_p}{\partial \omega_p} \mathbf{d}_m^H \mathbf{D} \mathbf{d}_p + i \frac{\partial \mathbf{u}_m}{\partial \omega_m} \mathbf{u}_p^H \mathbf{d}_m^H \mathbf{D} \mathbf{d}_p$$

and

$$r_N = \frac{1}{N} \sum_{t=1}^N \alpha_1^*(t) \alpha_2(t).$$

Using the fact that the polarization state vector of a COLD array is not a function of the direction parameter, thus $\partial \mathbf{u}_m / \partial \omega_m = \mathbf{0}$, consequently $k = 0$ and (3) becomes

$$[\mathbf{F}]_{mp} = N \Re \{ r_N \mathbf{u}_m^H \mathbf{u}_p \mathbf{d}_m^H \mathbf{D}^2 \mathbf{d}_p \}.$$

Furthermore, since the polarization state vector is normalized, one obtains

$$\mathbf{F} = N \begin{bmatrix} a_1^2 \alpha & \Re \{ r_N \mathbf{u}_1^H \mathbf{u}_2 \eta \} \\ \Re \{ r_N \mathbf{u}_1^H \mathbf{u}_2 \eta \} & a_2^2 \alpha \end{bmatrix}$$

where $\eta = \sum_{\ell=0}^{L-1} \ell^2 e^{i(\omega_1 - \omega_2)\ell}$, $\alpha = \frac{1}{6}(L-1)L(2L-1)$ and $\mathbf{u}_1^H \mathbf{u}_2 = \cos(\rho_1) \cos(\rho_2) + \sin(\rho_1) \sin(\rho_2) e^{i(\psi_2 - \psi_1)}$. Consequently, its inverse is given by

$$\mathbf{F}^{-1} = \frac{N}{\det\{\mathbf{F}\}} \begin{bmatrix} a_2^2 \alpha & -\Re\{r_N \mathbf{u}_1^H \mathbf{u}_2 \eta\} \\ -\Re\{r_N \mathbf{u}_1^H \mathbf{u}_2 \eta\} & a_1^2 \alpha \end{bmatrix} \quad (4)$$

where

$$\det\{\mathbf{F}\} = N^2 (a_1^2 a_2^2 \alpha^2 - \Re^2\{r_N \mathbf{u}_1^H \mathbf{u}_2 \eta\}).$$

Finally, replacing (2) and (4) in $\mathbf{CRB}(\boldsymbol{\xi}) = \mathbf{FIM}^{-1}(\boldsymbol{\xi})$, one obtains

$$\mathbf{CRB}(\omega_1) \triangleq [\mathbf{CRB}(\boldsymbol{\xi})]_{1,1} = \frac{\sigma^2}{2N} \frac{a_2^2 \alpha}{a_1^2 a_2^2 \alpha^2 - \Re^2\{r_N \mathbf{u}_1^H \mathbf{u}_2 \eta\}} \quad (5)$$

$$\mathbf{CRB}(\omega_2) \triangleq [\mathbf{CRB}(\boldsymbol{\xi})]_{2,2} = \frac{\sigma^2}{2N} \frac{a_1^2 \alpha}{a_1^2 a_2^2 \alpha^2 - \Re^2\{r_N \mathbf{u}_1^H \mathbf{u}_2 \eta\}} \quad (6)$$

$$\mathbf{CRB}(\omega_1, \omega_2) \triangleq [\mathbf{CRB}(\boldsymbol{\xi})]_{1,2} = -\frac{\sigma^2}{2N} \frac{\Re\{r_N \mathbf{u}_1^H \mathbf{u}_2 \eta\}}{a_1^2 a_2^2 \alpha^2 - \Re^2\{r_N \mathbf{u}_1^H \mathbf{u}_2 \eta\}} \quad (7)$$

The CRB is used as a benchmark to evaluate the efficiency of suboptimal unbiased estimators, however, it does not indicate the achievable SRL by such estimators. In the next section, we will make use of the derived CRBs (5), (6) and (7) to derive the SRL for the proposed model.

4. STATISTICAL RESOLUTION LIMIT

To resolve two sources, Smith [7] proposed the following criterion: *Two sources are resolvable if*

$$\begin{aligned} & \text{standard deviation of source separation} \\ & \leq \text{source separation} \end{aligned}$$

Consequently, Smith defined the SRL as the source separation at which the equality in the above inequality is achieved, in other words, he defined the SRL as the source separation that is equals to its own CRB.

4.1. Statistical resolution limit for a COLD-ULA

Having $\mathbf{CRB}(\boldsymbol{\xi})$, one can deduce $\mathbf{CRB}(\check{\boldsymbol{\xi}})$ by using the change of variable formula

$$\mathbf{CRB}(\check{\boldsymbol{\xi}}) = \frac{\partial \mathbf{g}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^T} \mathbf{CRB}(\boldsymbol{\xi}) \frac{\partial \mathbf{g}^T(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}, \quad (8)$$

where $\check{\boldsymbol{\xi}} = \mathbf{g}(\boldsymbol{\xi}) = [\delta_\omega^{(COLD)} \sigma^2]^T$, in which $\delta_\omega^{(COLD)} = |\omega_1 - \omega_2|$ and where the Jacobian matrix

$$\left[\frac{\partial \mathbf{g}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^T} \right]_{i,j} = \frac{\partial [\mathbf{g}(\boldsymbol{\xi})]_i}{\partial [\boldsymbol{\xi}]_j}.$$

Consequently,

$$\frac{\partial \mathbf{g}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^T} = \begin{bmatrix} \text{sgn}(\omega_1 - \omega_2) & -\text{sgn}(\omega_1 - \omega_2) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\text{sgn}(z) = \frac{z}{|z|}$ for $z \neq 0$. Without loss of generality, let us suppose that $\omega_1 > \omega_2$, thus

$$\frac{\partial \mathbf{g}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}^T} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (9)$$

Using the Jacobian matrix above and (8), one obtains

$$\begin{aligned} \mathbf{CRB}(\delta_\omega^{(COLD)}) & \triangleq [\mathbf{CRB}(\check{\boldsymbol{\xi}})]_{1,1} \\ & = \mathbf{CRB}(\omega_1) + \mathbf{CRB}(\omega_2) - 2\mathbf{CRB}(\omega_1, \omega_2). \end{aligned}$$

Consequently, the SRL² is defined as $\delta_\omega^{(COLD)}$ which resolve the following equation

$$\left(\delta_\omega^{(COLD)} \right)^2 = \mathbf{CRB}(\omega_1) + \mathbf{CRB}(\omega_2) - 2\mathbf{CRB}(\omega_1, \omega_2) \quad (10)$$

Consequently, we have to solve

$$\left(\delta_\omega^{(COLD)} \right)^2 = \frac{\sigma^2}{2N} \frac{(a_1^2 + a_2^2)\alpha + 2\Re\{r_N \mathbf{u}_1^H \mathbf{u}_2 \eta\}}{a_1^2 a_2^2 \alpha^2 - \Re^2\{r_N \mathbf{u}_1^H \mathbf{u}_2 \eta\}}. \quad (11)$$

4.1.1. The orthogonal sources case

In case of orthogonal sources ($r_N = 0$ [15]), the SRL for orthogonal sources is given by

$$\begin{aligned} \delta_\omega^{(COLD-O)} & = \frac{\sigma}{\sqrt{2N\alpha}} \sqrt{\frac{(a_1^2 + a_2^2)}{a_1^2 a_2^2}} \\ & = \frac{\sigma}{a_1 a_2} \sqrt{\frac{3(a_1^2 + a_2^2)}{NL(2L^2 - 3L + 1)}}. \end{aligned} \quad (12)$$

For orthogonal sources, it can be readily checked that the SRL is not a function of polarisation parameters. This is a surprising result. Note also that the SRL is proportional to the inverse of the third-half square-root of the number and to the square-root of sensors and amplitudes. Furthermore, the SRL obtained herein is, qualitatively, consistent with the SRL derived for binary phase-shift keying sources in [16], since it is proportional to the square-root of the variance.

4.1.2. The non-orthogonal sources case

Considering the first-order Taylor expansion of functional

$$\eta \stackrel{1}{=} \sum_{\ell=0}^{L-1} \ell^2 \left(1 + i\delta_\omega^{(COLD)} \ell \right) = \alpha + i\beta \delta_\omega^{(COLD)}$$

where

$$\beta = \sum_{\ell=0}^{L-1} \ell^3 = \frac{1}{4}(L-1)^2 L^2$$

²From (10), one should note that the SRL using the Smith criterion [7], unlike the Lee criterion [6], takes into account the correlation between sources.

thus expression (11) for non-orthogonal sources ($r_N \neq 0$) becomes

$$\left(\delta_\omega^{(COLD)}\right)^2 \stackrel{1}{=} \frac{\sigma^2}{2N} \frac{A + 2B - 2\delta_\omega^{(COLD)}\bar{B}}{C^2 - (B - \delta_\omega^{(COLD)}\bar{B})^2} \quad (13)$$

where $A = (a_1^2 + a_2^2)\alpha$, $B = \alpha\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\}$, $\bar{B} = \beta\Im\{r_N\mathbf{u}_1^H\mathbf{u}_2\}$, and $C = a_1a_2\alpha$ in which $\Im\{z\}$ denotes the imaginary part of z . Expression (13) is in fact the resolution of a fourth-order polynomial given by

$$\begin{aligned} & 2N\bar{B}\left(\delta_\omega^{(COLD)}\right)^4 + 4NB\bar{B}\left(\delta_\omega^{(COLD)}\right)^3 \\ & + 2N(B^2 - C^2)\left(\delta_\omega^{(COLD)}\right)^2 \\ & - 2\sigma^2\bar{B}\delta_\omega^{(COLD)} + \sigma^2(A + 2B) = 0. \end{aligned}$$

This leads to intractable solutions for the SRL. Only keeping the dominant terms lower or equal to the second-order, one obtains

$$\begin{aligned} & 2N(B^2 - C^2)\left(\delta_\omega^{(COLD)}\right)^2 - (2\sigma^2\bar{B})\delta_\omega^{(COLD)} \\ & + \sigma^2(A + 2B) = 0. \end{aligned}$$

The discriminant is given by $\Delta = 4\sigma^4\bar{B}^2 + 8\sigma^2N(C^2 - B^2)(A + 2B)$. Consequently, assuming that $\Delta \geq 0$, the solution is given by

$$\begin{aligned} \delta_\omega^{(COLD)} &= \frac{2\sigma^2\bar{B} \pm \sqrt{4\sigma^4\bar{B}^2 + 8\sigma^2N(C^2 - B^2)(A + 2B)}}{4N(B^2 - C^2)} \\ &= \frac{2\sigma^2\bar{B} \pm 2\sigma\sqrt{\sigma^2\bar{B}^2 + 2N(C^2 - B^2)(A + 2B)}}{4N(B^2 - C^2)} \\ &= \frac{\sigma^2\beta\Im\{r_N\mathbf{u}_1^H\mathbf{u}_2\} \pm \sigma\sqrt{h}}{2N\alpha^2(\Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\} - a_1^2a_2^2)}, \end{aligned}$$

where

$$\begin{aligned} h &= \sigma^2\beta^2\Im^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\} \\ &+ 2N\alpha^3(a_1^2a_2^2 - \Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\})((a_1^2 + a_2^2) + 2\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\}). \end{aligned}$$

Under **A3.**, the deterministic CRB is reachable only at high SNR [12], consequently, one can assume that σ^2 is small. In this case, this leads to the following positive solution

$$\delta_\omega^{(COLD)} \approx \frac{\sigma}{\sqrt{2N\alpha}} \sqrt{\frac{(a_1^2 + a_2^2) + 2\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\}}{a_1^2a_2^2 - \Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\}}} \quad (14)$$

Consequently, from (12) and (14), we notice that the SRL depends strongly on the state vector parameter, thus the performance of a COLD-ULA for orthogonal sources is better than for non-orthogonal sources, i.e.,

$$\delta_\omega^{(COLD-O)} < \delta_\omega^{(COLD)} \text{ iff } \Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\} > 0 \text{ or}$$

$$\frac{\Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\}}{a_1^2a_2^2} > 2 \frac{\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\}}{a_1^2 + a_2^2}.$$

Furthermore $\delta_\omega^{(COLD-O)} = \delta_\omega^{(COLD)}$ iff $\mathbf{u}_1^H\mathbf{u}_2 = 0$ meaning that the orthogonality of the vectors of the polarization state parameters induces the same performance regardless the orthogonality of sources.

4.2. Comparison between the statistical resolution limit of a COLD-ULA and a ULA

Consider two radiating far-field and narrowband sources observed by an ULA of L sensors with interelement spacing d [11]. The array and the emitted signals are coplanar. Furthermore, the additive noise and the model source are defined as in Section II. Following the same steps leading to $\delta_\omega^{(COLD-O)}$, one obtains after some algebraic calculations the SRL for the ULA denoted by $\delta_\omega^{(ULA-O)}$.

4.2.1. Comparison in the orthogonal sources case

In the case where the sources are orthogonal, one obtains $\delta_\omega^{(ULA)} = \delta_\omega^{(COLD)}$ meaning that, in the case of orthogonal sources, the performance of the COLD-ULA and the ULA are similar.

4.2.2. Comparison in the non-orthogonal sources case

In the case where the sources are non-orthogonal

$$\delta_\omega^{(ULA)} \approx \frac{\sigma}{\sqrt{2N\alpha}} \sqrt{\frac{(a_1^2 + a_2^2) + 2\Re\{r_N\}}{a_1^2a_2^2 - \Re^2\{r_N\}}} \quad (15)$$

Thus, from (14) and (15), one obtains

$$\frac{\delta_\omega^{(COLD)}}{\delta_\omega^{(ULA)}} \approx \sqrt{\frac{((a_1^2 + a_2^2) + 2\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\})(a_1^2a_2^2 - \Re^2\{r_N\})}{(a_1^2a_2^2 - \Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\})((a_1^2 + a_2^2) + 2\Re\{r_N\})}}$$

Which leads to the following implication

$$\delta_\omega^{(COLD)} < \delta_\omega^{(ULA)} \text{ iff } \Re\{r_N\} > \Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\} \quad (16)$$

Consequently, if the sources are non-orthogonal, one can distinguish the following cases

C1. if the amplitudes are positive reals, i.e., $\Im\{r_N\} = 0$, thus

$$\begin{aligned} \Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\} &= r_N\Re\{\mathbf{u}_1^H\mathbf{u}_2\} \\ &= r_N \sin(\phi_1) \sin(\phi_2) \sin(\psi_2 - \psi_1), \end{aligned}$$

so $\Re\{r_N\} > \Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\}$ and consequently $\delta_\omega^{(COLD)} < \delta_\omega^{(ULA)}$.

C2. if $\Re\{r_N\} > 0$ and $\psi_1 = \psi_2$ thus

$$\begin{aligned} \Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\} &= \Re\{r_N(\cos(\phi_1)\cos(\phi_2) + \sin(\phi_1)\sin(\phi_2))\} \\ &= \Re\{r_N\}(\cos(\phi_1)\cos(\phi_2) + \sin(\phi_1)\sin(\phi_2)) \\ &= \Re\{r_N\}\cos(\phi_1 - \phi_2), \text{ thus, } \Re\{r_N\} > \Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\} \\ &\text{and consequently } \delta_\omega^{(COLD)} < \delta_\omega^{(ULA)}. \end{aligned}$$

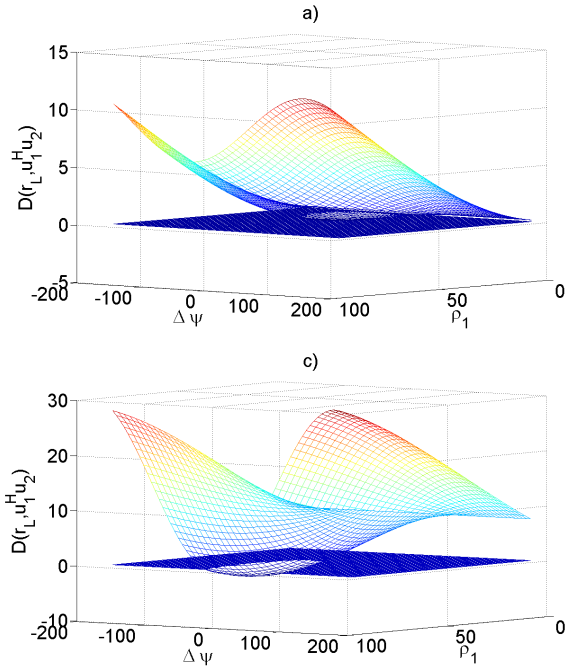


Fig. 1. $D(r_L, \mathbf{u}_1^H \mathbf{u}_2)$ Vs. the polarization state parameters ρ and ψ ; (a) $r_N = 9 + 15i$ and $\rho_2 = 10$, (c) $r_N = 15 + 9i$ and $\rho_2 = 60$.

C3. if the polarization state vectors are the same, i.e., $\rho_1 = \rho_2$ and $\psi_1 = \psi_2$, thus $\delta_\omega^{(COLD)} = \delta_\omega^{(ULA)}$ meaning that the use of an ULA is equivalent, in term of performance, to a COLD-ULA if the polarization of the sources is the same. This is expected since, *intuitively*, the same polarization does not bring additional information to resolve two sources.

Besides **C1.**, **C2.** and **C3.**, in Fig. 1 we plot

$$D(r_L, \mathbf{u}_1^H \mathbf{u}_2) = \Re\{r_N\} - \Re\{r_N \mathbf{u}_1^H \mathbf{u}_2\}$$

versus the polarization state parameters ρ and ψ . Consequently, from (16) if $D > 0$ thus $\delta_\omega^{(COLD)} < \delta_\omega^{(ULA)}$. Fig. 1 suggests that generally $\delta_\omega^{(COLD)} < \delta_\omega^{(ULA)}$ whereas $\delta_\omega^{(COLD)} > \delta_\omega^{(ULA)}$ only for a small region (which corresponds to the part of the plot that is under the horizontal plan), meaning that generally the performance of the COLD-ULA is better than the performance of the ULA.

5. CONCLUSION

In this paper, we derived the deterministic CRB in a non-matrix closed form expression for two polarized far-field time-varying narrowband known sources observed by a COLD-ULA. Taking advantage of these expressions, we deduced the SRL for the COLD-ULA which was compared to the SRL for the ULA. We noticed that, in the case where the sources are orthogonal, the SRL for the COLD-ULA is equal to the SRL for the ULA, meaning that it is not a function of polarisation parameters. This was not expected. Furthermore, for non-orthogonal sources, we gave

a sufficient and a necessary condition such that the SRL for the COLD-ULA is less than the SRL for the ULA. By analytical expressions and numerical simulations we showed that the SRL for the ULA is less than the SRL for the COLD-ULA only in few cases, meaning that generally the performance of the COLD-ULA is better than the performance of the ULA.

6. REFERENCES

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