STATISTICAL RESOLUTION LIMIT FOR SOURCE LOCALIZATION IN A MIMO CONTEXT

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ABSTRACT

In this paper, we derive the Multidimensional Statistical Resolution Limit (MSRL) to resolve two closely spaced targets using a widely spaced MIMO radar. Toward this end, we perform a hypothesis test formulation using the Generalized Likelihood Ratio Test (GLRT). More precisely, we link the MSRL to the minimum Signal-to-Noise Ratio (SNR) required to resolve two closely spaced targets, for a given probability of false alarm and for a given probability of detection. Finally, theoretical and numerical analysis of the MSRL are given for several scenarios (known/unknown parameters of interest and known/unknown noise variance) including lacunar arrays.

INDEX TERMS[1]

Multidimensional statistical resolution limit, MIMO radar, performance analysis.

1. INTRODUCTION

Based on the attractive Multi-Input Multi-Ouput (MIMO) communication theory, the MIMO radar has been received an increasing interest [1]. The advantage of the MIMO radar is the use of multiple antennas to simultaneously transmit several noncoherent known waveforms and exploits multiple antennas to receive the reflected signals (echoes).

One can find a plethora of algorithm for target localization using a MIMO radar and some related minimal bounds (see [1–4] and references therein). However their ultimate performance in terms of the Statistical Resolution Limit (SRL) has not been fully investigated. The SRL [5–9], defined as the minimal separation between two signals in terms of the parameter of interest allowing a correct source resolvability, is a challenging problem and an essential tool to quantify the estimator performance.

To the best of our knowledge, no results are available concerning the SRL for a MIMO radar with widely separated arrays (*i.e.,* where the transmitter and the receiver are far enough so that they do not share the same angle variable [2, 4]). The goal of this paper is to fill this lack. More precisely, the relationships between the Multidimensional SRL (MSRL) and the minimum SNR, required to resolve two closely spaced signal sources using a MIMO radar are investigated. The cases of known/unknown parameters of interest and known/unknown nuisance parameters are studied. With a similar methodology as [7], we perform a hypothesis test formulation (detection approach) using the Generalized Likelihood Ratio Test (GLRT). The choice of this strategy is motivated by the nice property of the GLRT (*i.e.,* it is an asymptotically Uniformly Most Powerful (UMP) test among all the invariant statistical tests [10]. This is the strongest statement of optimality that one could hope to obtain). Furthermore, in this paper, it is shown that the proposed test has the same behavior compared to the (ideal) clairvoyant detector in the Neyman-Pearson sense.

Consequently, in this paper, we derive closed form expressions of the MSRL in known/unknown parameters of interest and known/unknown nuisance parameters. Finally, theoretical and numerical analysis of the MSRL are given for several scenarios including lacunar arrays.

2. PROBLEM SETUP

2.1. Model setup

The output of a bistatic MIMO radar (in the case of widely spaced arrays with two targets) [4] is described for the ℓ-th pulse as follows:

$$\mathbf{X}_\ell = \sum_{m=1}^{2} \rho_m e^{2i\pi f_m \ell} \mathbf{a}_{\mathcal{R}}(\omega_m^{(\mathcal{R})}) \mathbf{a}_{\mathcal{T}}(\omega_m^{(\mathcal{T})})^T \mathbf{S} + \mathbf{W}_\ell, \;\; \ell \in [0 : L-1]$$

where L, ρ_m, f_m denote the number of samples per pulse period, a coefficient proportional to the Radar Cross-Section (RCS), the normalized Doppler frequency of the m-th target. Whereas, $\mathbf{a}_{\mathcal{T}}(.)$, $\mathbf{a}_{\mathcal{R}}(.)$, $\mathbf{S}$ and $\mathbf{W}_\ell$ denote the receiver steering vector, the transmitter receiver steering vector, the source matrix and the noise matrix for the ℓ-th pulse, respectively. The upper-script letter T denotes the transpose operator, whereas, upper/sub-script calligraphic letters $\mathcal{T}$ and $\mathcal{R}$ denote the transmitter and the receiver part, respectively. The i-th elements of the steering vectors are given by $[\mathbf{a}_{\mathcal{T}}(\omega_m^{(\mathcal{T})})]_i = e^{j\omega_m^{(\mathcal{T})} d_i^{(\mathcal{T})}}$ and $[\mathbf{a}_R(\omega_m^{(\mathcal{R})})]_i = e^{j\omega_m^{(\mathcal{R})} d_i^{(\mathcal{R})}}$ where $\omega_m^{(\mathcal{T})} = \frac{2\pi}{\nu}\sin(\psi_m)$ and $\omega_m^{(\mathcal{R})} = \frac{2\pi}{\nu}\sin(\theta_m)$ in which ψ_m is the angle of the target with respect to the transmit array (*i.e.*, DOD), θ_m is the angle of the target with respect to the reception array (*i.e.*, DOA), ν is the wavelength. The distance between a reference sensors (the first sensor herein) and the i-th sensor is denoted by $d_i^{(\mathcal{T})}$ and $d_i^{(\mathcal{R})}$ for the transmission and the reception arrays, respectively (*e.g.,* in the case of Uniform Linear Transmission Array (ULTA), $d_i^{(\mathcal{T})} = (i-1)d_{\mathcal{T}}$ where $d_{\mathcal{T}}$ is the inter-element space between two successive transmission sensors). The known source matrix is given by $\mathbf{S} = \begin{bmatrix}\mathbf{s}_0 & \dots & \mathbf{s}_{N_{\mathcal{T}}-1}\end{bmatrix}^T$ where $\mathbf{s}_{N_t} = \begin{bmatrix}s_{N_t}(1) & \dots & s_{N_t}(T)\end{bmatrix}^T$, in which $N_{\mathcal{T}}$ and T denote the number of transmission sensors and the number of snapshots, respectively. The diversity of the MIMO radar in terms of waveform coding allows to transmit orthogonal waveforms [2], *i.e.*,

[1]This project is funded by region Île de France and Digeteo Research Park.

$\mathbf{S}\mathbf{S}^H = \mathbf{S}^*\mathbf{S}^T = T\mathbf{I}_{N_\mathcal{T}}$. After matched filtering, one obtains $\mathbf{Y}_\ell = \frac{1}{\sqrt{T}}\mathbf{X}_\ell\mathbf{S}^H = \sum_{m=1}^{2}\alpha_m e^{2i\pi f_m \ell}\mathbf{a}_\mathcal{R}(\omega_m^{(\mathcal{R})})\mathbf{a}_\mathcal{T}(\omega_m^{(\mathcal{T})})^T + \mathbf{Z}_\ell$ where $\alpha_m = \sqrt{T}\rho_m$ and $\mathbf{Z}_\ell = \frac{1}{\sqrt{T}}\mathbf{W}_\ell\mathbf{S}^H$ denotes the noise matrix after the matched filtering. It is straightforward to rewrite the above matrix-based expression as a vectorized CanDecomp/Parafac [3, 11] model of dimension $P = 3$ according to

$$\mathbf{y} = [\text{vec}(\mathbf{Y}_0)^T \dots \text{vec}(\mathbf{Y}_{L-1})^T]^T = \mathbf{x} + \mathbf{z} \quad (1)$$

where $\mathbf{z} = [\mathbf{z}_0^T \dots \mathbf{z}_{L-1}^T]^T$ with $\mathbf{z}_\ell = \text{vec}(\mathbf{Z}_\ell)$ and

$$\mathbf{x} = \sum_{m=1}^{2}\alpha_m\left(\mathbf{c}(f_m)\otimes\mathbf{a}_\mathcal{T}(\omega_m^{(\mathcal{T})})\otimes\mathbf{a}_\mathcal{R}(\omega_m^{(\mathcal{R})})\right) \quad (2)$$

in which $\mathbf{c}(f_m) = [1\ e^{2i\pi f_m}\ \dots\ e^{2i\pi f_m(L-1)}]^T$ and where $\otimes$ denotes the Kronecker product.

2.2. Statistic of the observation

Assuming that the complex Gaussian noise interferences (before the matched filtering) are independent and identically distributed (IID) samples with zero-mean and a covariance matrix $\sigma^2\mathbf{I}$ [1] (the clutter and jammer echoes are not considered in this work) and thanks to the orthogonality of the waveforms, one can notice that $E(\mathbf{z}_\ell\mathbf{z}_\ell^H) = \frac{1}{T}(\mathbf{S}^*\otimes\mathbf{I}_{N_\mathcal{R}})E(\text{vec}(\mathbf{W}_\ell)\text{vec}(\mathbf{W}_{\ell'})^H)(\mathbf{S}^T\otimes\mathbf{I}_{N_\mathcal{R}}) = \sigma^2\mathbf{I}_{N_\mathcal{T}N_\mathcal{R}}$ and $E(\mathbf{z}_\ell\mathbf{z}_{\ell'}^H) = \mathbf{0}$ for $\ell \neq \ell'$ in which $N_\mathcal{R}$ denotes the number of receiver sensors. Thus, $E(\mathbf{z}\mathbf{z}^H) = \sigma^2\mathbf{I}_{LN_\mathcal{T}N_\mathcal{R}}$. Consequently, the observation follows a complex Gaussian distribution according to $\mathbf{y}\sim\mathcal{CN}(\mathbf{x}, \sigma^2\mathbf{I}_{LN_\mathcal{T}N_\mathcal{R}})$.

2.3. Assumptions

Throughout the rest of the paper, the following assumptions are assumed to hold: $\mathcal{A}$**1)** The signal sources and the array geometry are known. $\mathcal{A}$**2)** For sake of simplicity the Doppler frequencies are assumed to be equal $f_1 = f_2 = f$ (or even null). Nevertheless, numerical simulations will show that the derived MSRL (with equal Doppler frequency assumption) has the same behavior compared to the clairvoyant detector. $\mathcal{A}$**3)** Finally, we consider α_1, α_2 as unknown unequal deterministic parameters (note that both case of known and unknown σ^2 are studied in the remaining of the paper.)

3. DETECTION APPROACH

3.1. Hypothesis test formulation

Resolving two closely spaced sources, with respect to their parameter of interest $\omega_m^{(\mathcal{T})}$ and $\omega_m^{(\mathcal{R})}$, can be formulated as a binary hypothesis test [7, 8]. The hypothesis $\mathcal{H}_0$ represents the case where the two emitted signal sources are combined onto one signal (*i.e.,* it represents the case of two unresolvable targets), whereas the hypothesis $\mathcal{H}_1$ embodies the situation where the two signals are resolvable. Thus, one obtains the following binary hypothesis test:

$$\begin{cases}\mathcal{H}_0: & (\delta_\mathcal{R}, \delta_\mathcal{T}) = (0,0),\\ \mathcal{H}_1: & (\delta_\mathcal{R}, \delta_\mathcal{T}) \neq (0,0),\end{cases} \quad (3)$$

where the so-called Local SRLs (LSRL) are given by $\delta_\mathcal{T} \triangleq \omega_2^{(\mathcal{T})} - \omega_1^{(\mathcal{T})}$ and $\delta_\mathcal{R} \triangleq \omega_2^{(\mathcal{R})} - \omega_1^{(\mathcal{R})}$. Since the LSRLs are unknown, it is impossible to design an optimal detector in the Neyman-Pearson sense. Alternatively, the Generalized Likelihood Ratio Test (GLRT) statistic [10] is a well known approach appropriate to solve such a problem. The GLRT statistic is expressed as $G(\mathbf{y}) = \frac{p(\mathbf{y};\hat{\delta}_\mathcal{R},\hat{\delta}_\mathcal{T},\hat{\boldsymbol{\rho}}_1,\mathcal{H}_1)}{p(\mathbf{y};\hat{\boldsymbol{\rho}}_0,\mathcal{H}_0)} \gtrless_{\mathcal{H}_0}^{\mathcal{H}_1} \eta'$, in which $p(\mathbf{y};\hat{\boldsymbol{\rho}}_0,\mathcal{H}_0)$ and $p(\mathbf{y};\hat{\delta}_\mathcal{R},\hat{\delta}_\mathcal{T},\hat{\boldsymbol{\rho}}_1,\mathcal{H}_1)$ denote the probability density function of the observation under $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Where η', $\hat{\delta}_\mathcal{R}$, $\hat{\delta}_\mathcal{T}$ and $\hat{\boldsymbol{\rho}}_i$ denote the detection threshold, the Maximum Likelihood Estimate (MLE) of $\delta_\mathcal{R}$ and $\delta_\mathcal{T}$ under $\mathcal{H}_1$ and the MLE of the parameter vector $\boldsymbol{\rho}_i$ (containing all the unknown nuisance and/or unwanted parameters) under $\mathcal{H}_i, i = 0, 1$.

One can easily see that the derivation of $\hat{\delta}_\mathcal{R}$ and $\hat{\delta}_\mathcal{T}$ is a nonlinear optimization problem, which is analytically intractable. Using the fact that the separation is small (this assumption can be argued by the fact that the high resolution algorithms have asymptotically an infinite resolving power [12]), one can approximate the model (2) into a model which is linear w.r.t. the unknown parameters.

3.2. Linear form of the MIMO model

First, let us introduce the so-called center parameters $\omega_c^{(\mathcal{T})} \triangleq \frac{\omega_1^{(\mathcal{T})}+\omega_2^{(\mathcal{T})}}{2}$ and $\omega_c^{(\mathcal{R})} \triangleq \frac{\omega_1^{(\mathcal{R})}+\omega_2^{(\mathcal{R})}}{2}$. Second, using the first order Taylor expansion around $\delta_\mathcal{T} = 0$ and $\delta_\mathcal{R} = 0$ of (2), one obtains $\mathbf{a}_\mathcal{T}(\omega_1^{(\mathcal{T})}) = \mathbf{a}_\mathcal{T}(\omega_c^{(\mathcal{T})}) - \frac{j}{2}\delta_\mathcal{T}\dot{\boldsymbol{a}}_\mathcal{T}(\omega_c^{(\mathcal{T})})$, $\mathbf{a}_\mathcal{R}(\omega_1^{(\mathcal{R})}) = \mathbf{a}_\mathcal{R}(\omega_c^{(\mathcal{R})}) - \frac{j}{2}\delta_\mathcal{R}\dot{\boldsymbol{a}}_\mathcal{R}(\omega_c^{(\mathcal{R})})$, $\mathbf{a}_\mathcal{T}(\omega_2^{(\mathcal{T})}) = \mathbf{a}_\mathcal{T}(\omega_c^{(\mathcal{T})}) + \frac{j}{2}\delta_\mathcal{T}\dot{\boldsymbol{a}}_\mathcal{T}(\omega_c^{(\mathcal{T})})$ and $\mathbf{a}_\mathcal{R}(\omega_2^{(\mathcal{R})}) = \mathbf{a}_\mathcal{R}(\omega_c^{(\mathcal{R})}) + \frac{j}{2}\delta_\mathcal{R}\dot{\boldsymbol{a}}_\mathcal{R}(\omega_c^{(\mathcal{R})})$, where $\dot{\boldsymbol{a}}_\mathcal{T}(.) \triangleq \boldsymbol{a}_\mathcal{T}(.)\odot\boldsymbol{d}_\mathcal{T}$, and $\dot{\boldsymbol{a}}_\mathcal{R}(.) \triangleq \boldsymbol{a}_\mathcal{R}(.)\odot\boldsymbol{d}_\mathcal{R}$ in which $\boldsymbol{d}_T = [d_0^{(\mathcal{T})}\ d_1^{(\mathcal{T})}\ \dots\ d_{N-1}^{(\mathcal{T})}]^T$, $\boldsymbol{d}_R = [d_0^{(\mathcal{R})}\ d_1^{(\mathcal{R})}\ \dots\ d_{N-1}^{(\mathcal{R})}]^T$ and $\odot$ denoting the Hadamard products. Thus, one can approximate (1) by the following expression

$$\mathbf{y} = \boldsymbol{G}\boldsymbol{\zeta} + \mathbf{z}, \quad (4)$$

where the $(LN_\mathcal{T}N_\mathcal{R})\times 4$ matrix $\boldsymbol{G}$ is defined as $\boldsymbol{G} = \begin{bmatrix}\boldsymbol{\varrho}_1 & \boldsymbol{\varrho}_2 & \boldsymbol{\varrho}_3 & \boldsymbol{\varrho}_4\end{bmatrix}$ in which $\boldsymbol{\varrho}_1 = \mathbf{c}(f)\otimes\boldsymbol{a}_\mathcal{T}(\omega_c^{(\mathcal{T})})\otimes\boldsymbol{a}_\mathcal{R}(\omega_c^{(\mathcal{R})})$, $\boldsymbol{\varrho}_2 = \mathbf{c}(f)\otimes\boldsymbol{a}_\mathcal{T}(\omega_c^{(\mathcal{T})})\otimes\dot{\boldsymbol{a}}_\mathcal{R}(\omega_c^{(\mathcal{R})})$, $\boldsymbol{\varrho}_3 = \mathbf{c}(f)\otimes\dot{\boldsymbol{a}}_\mathcal{T}(\omega_c^{(\mathcal{T})})\otimes\boldsymbol{a}_\mathcal{R}(\omega_c^{(\mathcal{R})})$ and $\boldsymbol{\varrho}_4 = \mathbf{c}(f)\otimes\dot{\boldsymbol{a}}_\mathcal{T}(\omega_c^{(\mathcal{T})})\otimes\dot{\boldsymbol{a}}_\mathcal{R}(\omega_c^{(\mathcal{R})})$. The unknown 4×1 parameter vector is given by

$$\boldsymbol{\zeta} = \begin{bmatrix}\alpha_1+\alpha_2\\ \frac{j}{2}\delta_\mathcal{R}(\alpha_2-\alpha_1)\\ \frac{j}{2}\delta_\mathcal{T}(\alpha_2-\alpha_1)\\ \frac{-1}{4}\delta_\mathcal{R}\delta_\mathcal{T}(\alpha_1+\alpha_2)\end{bmatrix}. \quad (5)$$

In the remaining of this paper, the parameters $\omega_c^{(\mathcal{T})}$ and $\omega_c^{(\mathcal{R})}$ (which represent the center parameters) are assumed to be known [8] or previously estimated [7]. In the following, we use the linear form of the signal model (4). Both cases of known and unknown noise variance will be considered.

4. DERIVATION AND ANALYSIS OF THE MSRL

4.1. Case of known noise variance

Using the linear form in (4), the binary hypothesis test in (3) can be re-formulated as follows

$$\begin{cases}\mathcal{H}_0: & \boldsymbol{P}\boldsymbol{\zeta} = \mathbf{0},\\ \mathcal{H}_1: & \boldsymbol{P}\boldsymbol{\zeta} \neq \mathbf{0},\end{cases} \quad (6)$$

where $\boldsymbol{P} = \begin{bmatrix}\mathbf{0} & \mathbf{I}_3\end{bmatrix}$ is a selection matrix and where $\boldsymbol{\rho}$ is reduced to an empty vector. Note that the test (6) is connected to the so-called Multidimensional SRL (MSRL), defined as $\boldsymbol{\delta} \triangleq [\delta_\mathcal{R}\ \delta_\mathcal{T}\ \delta_\mathcal{T}\delta_\mathcal{R}]^T$ according to $\boldsymbol{P}\boldsymbol{\zeta} = \boldsymbol{Q}\boldsymbol{\delta}$ in which $\boldsymbol{Q} = \text{diag}\{\frac{j}{2}(\alpha_2-\alpha_1), \frac{j}{2}(\alpha_2-\alpha_1), \frac{-1}{4}(\alpha_1+\alpha_2)\}$. The hypothesis test (6) is a detection problem of a deterministic signals in unknown parameters and known noise variance [10] where the GLRT statistic yields to $T_\text{K}(\mathbf{y}) = \frac{2}{\sigma^2}\hat{\boldsymbol{\zeta}}^H\boldsymbol{P}^T\left(\boldsymbol{P}\left(\boldsymbol{G}^H\boldsymbol{G}\right)^{-1}\boldsymbol{P}^T\right)^{-1}\boldsymbol{P}\hat{\boldsymbol{\zeta}} \gtrless_{\mathcal{H}_0}^{\mathcal{H}_1} \eta_\text{K}$ where the subscript K stands for the case of Known noise variance. The MLE of $\boldsymbol{\zeta}$ is given by $\hat{\boldsymbol{\zeta}} = \boldsymbol{G}^{\ddagger}\mathbf{y}$ where $\boldsymbol{G}^{\ddagger}$, the pseudo inverse matrix, is given by $\boldsymbol{G}^{\ddagger} = \left(\boldsymbol{G}^H\boldsymbol{G}\right)^{-1}\boldsymbol{G}^H$. The value of $\eta_{\rm K}$ is conditioned by the choice of the probability of false alarm P_{fa} and the probability of detection P_d. The performance of the latter hypothesis test is characterized by $P_{fa} = \mathcal{Q}_{\chi_6^2}(\eta_{\rm K})$ and $P_d = \mathcal{Q}_{\chi_6^2(\lambda_{\rm K}(P_{fa},P_d))}(\eta_{\rm K})$ [10] where χ_6^2 and $\chi_6^2(\lambda_{\rm K}(P_{fa},P_d))$ denote the central and the non-central chi-square distribution of 6 degrees of freedom[2], respectively, in which $\mathcal{Q}_{\chi_6^2}(.)$ and $\mathcal{Q}_{\chi_6^2(\lambda(P_{fa},P_d))}(.)$ denote the right tail of the pdf χ_6^2 and the pdf $\chi_6^2(\lambda(P_{fa},P_d))$, respectively. Furthermore, the non-centrality parameter is given by

$$\lambda_{\rm K}(P_{fa},P_d) = \frac{2}{\sigma^2}\boldsymbol{\delta}^T\boldsymbol{Q}^*\left(\boldsymbol{P}\left(\boldsymbol{G}^H\boldsymbol{G}\right)^{-1}\boldsymbol{P}^T\right)^{-1}\boldsymbol{Q}\boldsymbol{\delta}. \quad (7)$$

On the other hand, one should notice that $\lambda_{\rm K}(P_{fa},P_d)$ can be derived for a given P_{fa} and P_d as the solution of $\mathcal{Q}^{-1}_{\chi_2^2}(P_{fa}) = \mathcal{Q}^{-1}_{\chi_2^2(\lambda(P_{fa},P_d))}(P_d)$ [8]. Consequently, one obtains

$$\text{SNR}_{\rm K} \triangleq \frac{\text{trace}\left\{\boldsymbol{S}\boldsymbol{S}^H\right\}}{T\sigma^2} = \frac{N_{\mathcal{T}}\lambda_{\rm K}(P_{fa},P_d)}{2\boldsymbol{\delta}^T\boldsymbol{Q}^*\left(\boldsymbol{P}\left(\boldsymbol{G}^H\boldsymbol{G}\right)^{-1}\boldsymbol{P}^T\right)^{-1}\boldsymbol{Q}\boldsymbol{\delta}}. \quad (8)$$

To simplify (8), one should note that $\boldsymbol{G}^H\boldsymbol{G} = L\begin{bmatrix} N_{\mathcal{T}}N_{\mathcal{R}} & \boldsymbol{\kappa}^T \\ \boldsymbol{\kappa} & \boldsymbol{\Phi}\end{bmatrix}$ since $\|\boldsymbol{\varrho}_1\|^2 = LN_{\mathcal{T}}N_{\mathcal{R}}$ and $\boldsymbol{\Phi} = \begin{bmatrix} f_{0,2} & f_{1,1} & f_{1,2} \\ f_{1,1} & f_{2,0} & f_{2,1} \\ f_{1,2} & f_{2,1} & f_{2,2}\end{bmatrix}$, $\boldsymbol{\kappa} = \begin{bmatrix} f_{0,1} & f_{1,0} & f_{1,1}\end{bmatrix}^H$, where $f_{p,q} = \sum_{n_t=1}^{N_{\mathcal{T}}}\left(d_{n_t}^{(\mathcal{T})}\right)^p\sum_{n_r=1}^{N_{\mathcal{R}}}\left(d_{n_r}^{(\mathcal{R})}\right)^q$. Using the inversion lemma [13], one obtains $\left(\boldsymbol{G}^H\boldsymbol{G}\right)^{-1} = \frac{1}{L}\begin{bmatrix} \frac{1}{\beta} & -\frac{1}{\beta}\boldsymbol{\kappa}^T\boldsymbol{\Phi}^{-1} \\ -\frac{1}{\beta}\boldsymbol{\Phi}^{-1}\boldsymbol{\kappa} & \boldsymbol{\Phi}^{-1} + \frac{1}{\beta}\boldsymbol{\Phi}^{-1}\boldsymbol{\kappa}\boldsymbol{\kappa}^T\boldsymbol{\Phi}^{-1}\end{bmatrix}$ where the Schur complement is $\beta = N_{\mathcal{T}}N_{\mathcal{R}} - \boldsymbol{\kappa}^T\boldsymbol{\Phi}^{-1}\boldsymbol{\kappa}$. Multiplying $\left(\boldsymbol{G}^H\boldsymbol{G}\right)^{-1}$ by $\boldsymbol{P}$ on the left and by $\boldsymbol{P}^T$ on the right has the effect to eliminate the first column and the first row of $\left(\boldsymbol{G}^H\boldsymbol{G}\right)^{-1}$. Thus, $\left(\boldsymbol{P}\left(\boldsymbol{G}^H\boldsymbol{G}\right)^{-1}\boldsymbol{P}^T\right)^{-1} = \frac{1}{L}\left(\boldsymbol{\Phi}^{-1} + \frac{1}{\beta}\boldsymbol{\Phi}^{-1}\boldsymbol{\kappa}\boldsymbol{\kappa}^T\boldsymbol{\Phi}^{-1}\right)^{-1}$. Consequently, using the Woodbury formula [13], one obtains

$$\left(\boldsymbol{P}\left(\boldsymbol{G}^H\boldsymbol{G}\right)^{-1}\boldsymbol{P}^T\right)^{-1} = L\left(\boldsymbol{\Phi} - \frac{1}{N_{\mathcal{T}}N_{\mathcal{R}}}\boldsymbol{\kappa}\boldsymbol{\kappa}^T\right). \quad (9)$$

Denoting $\boldsymbol{K} \triangleq \frac{1}{N_{\mathcal{T}}}\left(\boldsymbol{\Phi} - \frac{1}{N_{\mathcal{T}}N_{\mathcal{R}}}\boldsymbol{\kappa}\boldsymbol{\kappa}^T\right)$ and plugging (9) into (8), one obtains:

Result 1 *The relationship between the MSRL $\boldsymbol{\delta}$ and the minimum SNR, required to resolve two closely spaced sources, is then given by*

$$SNR_{\rm K} = \frac{\lambda_{\rm K}(P_{fa},P_d)}{2L\boldsymbol{\delta}^T\boldsymbol{Q}^*\boldsymbol{K}\boldsymbol{Q}\boldsymbol{\delta}}. \quad (10)$$

4.2. Case of unknown noise variance

One can extend the latter analysis to the case of unknown noise variance σ^2 (*i.e.,* $\boldsymbol{\rho}$ is reduced to the scalar σ^2). The binary hypothesis test becomes then

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{P}\boldsymbol{\zeta} = \boldsymbol{0} \text{ with } \sigma^2 \text{ unknown}, \\ \mathcal{H}_1: & \boldsymbol{P}\boldsymbol{\zeta} \neq \boldsymbol{0} \text{ with } \sigma^2 \text{ unknown}.\end{cases} \quad (11)$$

[2]Since, $\boldsymbol{P}\boldsymbol{\zeta} \in \mathbb{C}^{3\times 1}$, thus the degree of freedom of the considered chi-squared pdf is equal to 6 instead of 3 in the real case.

The hypothesis test formulated in (11) is a detection problem of a deterministic signals in unknown parameters and unknown noise variance [10]. Its GLRT statistic is given by $T_{\rm U}(\mathbf{y}) = \frac{\mathbf{y}^H\left(\boldsymbol{G}^{\ddagger}\right)^H\boldsymbol{P}^T\left(\boldsymbol{P}\left(\boldsymbol{G}^H\boldsymbol{G}\right)^{-1}\boldsymbol{P}^T\right)^{-1}\boldsymbol{P}\boldsymbol{G}^{\ddagger}\mathbf{y}}{\mathbf{y}^H\boldsymbol{P}_{\boldsymbol{G}}^{\perp}\mathbf{y}} \gtrless_{\mathcal{H}_0}^{\mathcal{H}_1} \eta_{\rm U}$, in which the subscript U stands for Unknown noise variance and where $\boldsymbol{P}_{\boldsymbol{G}}^{\perp} = \boldsymbol{I} - \boldsymbol{G}\boldsymbol{G}^{\ddagger}$ denotes the orthogonal projection matrix. The performance of the later hypothesis test is characterized by $P_{fa} = \mathcal{Q}_{F_{6,LN_{\mathcal{T}}N_{\mathcal{R}}-6}}(\eta_{\rm U})$ and $P_d = \mathcal{Q}_{F_{6,LN_{\mathcal{T}}N_{\mathcal{R}}-6}(\lambda_{\rm U}(P_{fa},P_d))}(\eta_{\rm U})$ [10], where $F_{6,LN_{\mathcal{T}}N_{\mathcal{R}}-6}$ and $F_{6,LN_{\mathcal{T}}N_{\mathcal{R}}-6}(\lambda_{\rm U}(P_{fa},P_d))$ denote the central and the non-central F distribution with 6 and $LN_{\mathcal{T}}N_{\mathcal{R}}-6$ degree of freedom, respectively. The non-centrality parameter is given by[3]

$$\lambda_{\rm U}(P_{fa},P_d) = \frac{2}{\sigma^2}\boldsymbol{\zeta}^H\boldsymbol{P}^T\left(\boldsymbol{P}\left(\boldsymbol{G}^H\boldsymbol{G}\right)^{-1}\boldsymbol{P}^T\right)^{-1}\boldsymbol{P}\boldsymbol{\zeta}. \quad (12)$$

Note that, $\lambda_{\rm U}(P_{fa},P_d)$ can be derived for a given P_{fa} and P_d as the solution of $\mathcal{Q}^{-1}_{F_{6,LN_{\mathcal{T}}N_{\mathcal{R}}-6}}(P_{fa}) = \mathcal{Q}^{-1}_{F_{6,LN_{\mathcal{T}}N_{\mathcal{R}}-6}(\lambda(P_{fa},P_d))}(P_d)$ [8]. Thus, using (9) and (12) one has:

Result 2 *The relationship between the MSRL $\boldsymbol{\delta}$ and the minimum SNR, required to resolve two closely spaced sources with unknown noise variance, is then given by*

$$SNR_{\rm U} = \frac{\lambda_{\rm U}(P_{fa},P_d)}{2L\boldsymbol{\delta}^T\boldsymbol{Q}^*\boldsymbol{K}\boldsymbol{Q}\boldsymbol{\delta}}. \quad (13)$$

4.3. The ideal (clairvoyant) detector

In Result 1 and 2 we have derived the MSRL using the GLRT (recall that the Neyman-Pearson test cannot be conducted due to the fact that $\boldsymbol{\delta}$ is an unknown parameter). Thus, it is interesting to compare $\text{SNR}_{\rm K}$ and $\text{SNR}_{\rm U}$ with the SNR associated with the clairvoyant Neyman-Pearson test (where all the parameter are known even $\boldsymbol{\delta}$). Toward this aim, one can consider the new observation $\mathbf{y}' \triangleq \mathbf{y} - (\alpha_1+\alpha_2)\mathbf{c}(f)\otimes\boldsymbol{a}_{\mathcal{T}}(\omega_c^{(\mathcal{T})})\otimes\boldsymbol{a}_{\mathcal{R}}(\omega_c^{(\mathcal{R})})$. Thus, it can be shown that $\mathbf{y}' = \boldsymbol{G}\boldsymbol{P}^T\boldsymbol{P}\boldsymbol{\zeta} + \mathbf{z} = \boldsymbol{G}\boldsymbol{P}^T\boldsymbol{Q}\boldsymbol{\delta} + \mathbf{z}$, leading to the following binary hypothesis test

$$\begin{cases} \mathcal{H}_0: & \mathbf{y}' = \mathbf{z}, \\ \mathcal{H}_1: & \mathbf{y}' = \boldsymbol{G}\boldsymbol{P}^T\boldsymbol{Q}\boldsymbol{\delta} + \mathbf{z}.\end{cases} \quad (14)$$

The hypothesis test in (14) is a detection problem of a known deterministic signal in a known variance complex white Gaussian noise, which is a mean-shifted Gauss-Gauss detection problem such that $T_{\rm C}(\mathbf{y}') \sim \begin{cases}\mathcal{H}_0: \mathcal{CN}(0, \frac{\sigma^2\mathcal{E}}{2}) \\ \mathcal{H}_1: \mathcal{CN}(\mathcal{E}, \frac{\sigma^2\mathcal{E}}{2})\end{cases}$ [10], where the subscript C stands for the Clairvoyant case, in which $\mathcal{E} = \frac{2}{\sigma^2}\boldsymbol{\delta}^T\boldsymbol{Q}^*\boldsymbol{P}\boldsymbol{G}^H\boldsymbol{G}\boldsymbol{P}^T\boldsymbol{Q}\boldsymbol{\delta} = \frac{2}{\sigma^2}\boldsymbol{\delta}^T\boldsymbol{Q}^*\boldsymbol{\Phi}\boldsymbol{Q}\boldsymbol{\delta}$. On the other hand, the detection performance are given by $\lambda_{\rm C}(P_{fa},P_d) = \left(\mathcal{Q}^{-1}(P_{fa}) - \mathcal{Q}^{-1}(P_d)\right)^2$, in which $\lambda_{\rm C}$ denotes the so-called deflection coefficient, whereas $\mathcal{Q}^{-1}(.)$ is the inverse of the right-tail of probability function for a Gaussian random variable with zero mean and unit variance. Consequently, denoting $\boldsymbol{K}' = \frac{1}{N_{\mathcal{T}}}\boldsymbol{\Phi}$, one has

Result 3 *The relationship between the MSRL $\boldsymbol{\delta}$ and the minimum SNR, required to resolve two closely spaced sources in the optimal*

[3]Note that for the same P_{fa} and P_d, $\lambda_{\rm K}(P_{fa},P_d) \neq \lambda_{\rm U}(P_{fa},P_d)$. Meaning that for the same P_{fa} and P_d the noise variance $\sigma^2$and/or the MSRL will differ in the known/unknown variance case (see (7) and (12)).

(clairvoyant) case, is then given by

$$\text{SNR}_{\mathrm{C}} = \frac{\lambda_{\mathrm{C}}(P_{fa}, P_d)}{2L\boldsymbol{\delta}^T \boldsymbol{Q}^* \boldsymbol{K}' \boldsymbol{Q} \boldsymbol{\delta}}. \tag{15}$$

5. ANALYSIS OF THE MSRL

This section is devoted to the theoretical and numerical analysis of the MSRL (or equivalently their corresponding minimal SNRs).

- First, let us compare the derived SNR in i) the clairvoyant case, ii) the unknown parameters with known noise variance case and iii) the unknown parameters with unknown noise variance case. On one hand, from (10), (13) and (15) one obtains

$$\frac{\mathrm{SNR}_{\mathrm{C}}}{\mathrm{SNR}_{\mathrm{K}}} = \rho \frac{\lambda_{\mathrm{C}}(P_{fa}, P_d)}{\lambda_{\mathrm{K}}(P_{fa}, P_d)} \text{ where } \rho = \frac{\boldsymbol{\delta}^T \boldsymbol{Q}^* \boldsymbol{K} \boldsymbol{Q} \boldsymbol{\delta}}{\boldsymbol{\delta}^T \boldsymbol{Q}^* \boldsymbol{K}' \boldsymbol{Q} \boldsymbol{\delta}} \tag{16}$$

and

$$\frac{\mathrm{SNR}_{\mathrm{K}}}{\mathrm{SNR}_{\mathrm{U}}} = \frac{\lambda_{\mathrm{K}}(P_{fa}, P_d)}{\lambda_{\mathrm{U}}(P_{fa}, P_d)}. \tag{17}$$

On the other hand, note that: $\boldsymbol{\mathcal{P}1}$) for any $P_d > P_{fa}$ one has $\lambda_{\mathrm{C}}(P_{fa}, P_d) < \lambda_{\mathrm{K}}(P_{fa}, P_d) < \lambda_{\mathrm{U}}(P_{fa}, P_d)$ [7], $\boldsymbol{\mathcal{P}2}$) the Hermitian matrix $\boldsymbol{\Omega} = \boldsymbol{K}' - \boldsymbol{K} = \boldsymbol{\kappa}_0 \boldsymbol{\kappa}_0^H$ where $\boldsymbol{\kappa}_0 = \boldsymbol{Q}^* \boldsymbol{\kappa} / \sqrt{N_{\mathcal{T}} N_{\mathcal{R}}}$ is a positive semi-definite matrix. Thus, $\rho \leq 1$. Consequently, from (16), (17), $\boldsymbol{\mathcal{P}1}$ and $\boldsymbol{\mathcal{P}2}$ one deduces, as expected, that for fixed P_{fa} and P_d (such that $P_d > P_{fa}$) one has $\mathrm{SNR}_{\mathrm{C}} < \mathrm{SNR}_{\mathrm{K}} < \mathrm{SNR}_{\mathrm{U}}$. In Fig. 1 we have reported the LSRL w.r.t. $\delta_{\mathcal{R}}$ in the clairvoyant, the known noise variance and the unknown noise variance cases versus the SNR (the same conclusion are done also for the LSRL w.r.t. $\delta_{\mathcal{T}}$). One can notice that the LSRLs derived in the case of known and unknown noise variance cases have the same behavior than the one in the clairvoyant case. For the same MSRL (*i.e.,* for a fixed $\delta_{\mathcal{T}}$ and $\delta_{\mathcal{R}}$), the gap between $\mathrm{SNR}_{\mathrm{K}}$ and $\mathrm{SNR}_{\mathrm{U}}$ is exclusively due to the non-centrality parameters $\lambda_{\mathrm{K}}(P_{fa}, P_d)$ and $\lambda_{\mathrm{U}}(P_{fa}, P_d)$. This gap is approximatively equal to 1dB. Whereas, the gap between $\mathrm{SNR}_{\mathrm{C}}$ and $\mathrm{SNR}_{\mathrm{K}}$ is due to both: i) the ration of the deflection coefficient $\lambda_{\mathrm{C}}(P_{fa}, P_d)$ over the non-centrality parameter $\lambda_{\mathrm{K}}(P_{fa}, P_d)$, and, ii) the norm of $\boldsymbol{\Omega}$ which reflects the value of ρ. This latter gap, is evaluated to 9 dB.

- Second, the effect of missing sensors is considered herein. Let us consider different scenarios. In each scenario we have the same transmitter ULA with $N_{\mathcal{T}} = 10$ sensors but different receiver arrays (from a scenario to an other) having the same array aperture. Let us denote these receiver arrays by $A_{N_{\mathcal{R}}}$ where $N_{\mathcal{R}}$ represents the number of sensors in the lacunar receiver arrays. In Fig. 2 we plot the LSRL for the receiver (*i.e.,* we focus only on $\delta_{\mathcal{R}}$, the case of $\delta_{\mathcal{T}}$ has the same behavior) for different $A_{N_{\mathcal{R}}}$ with $N_{\mathcal{R}} \in \{5, 7, 8, 9, 10\}$. This figure represents qualitatively the loss due to a missing sensors (but for the same array aperture) which is evaluated to 3dB.

6. CONCLUSION

In this paper, we have derived the Multidimensional Statistical Resolution Limit (MSRL) for two closely spaced targets using a widely-spaced MIMO radar (made from possibly non-uniform/lacunar transmitter and receiver arrays). Toward this goal, we have conduct a hypothesis test approach. More precisely, we have use the Generalized Likelihood Ratio Test (GLRT). This analysis provides useful information concerning the behavior of the MSRL and the minimum SNR required to resolve two closely spaced targets for a given probability of false alarm and a given probability of detection. Finally, numerical simulations shows that the derived MSRL has the same behavior compared to the clairvoyant (ideal) detector.

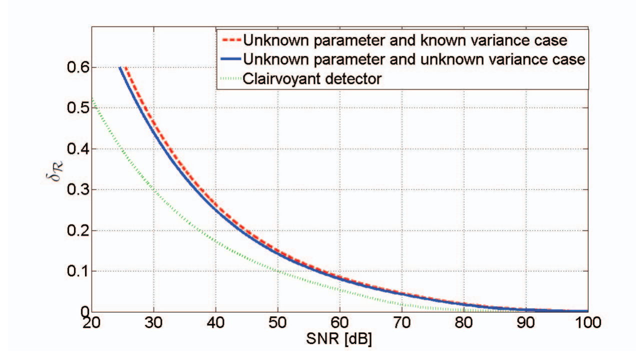

Fig. 1. The LSRL versus the required SNR to resolve two closely targets for $L = 4$, a transmitter and a receiver ULA with $N_{\mathcal{T}} = N_{\mathcal{R}} = 4$ and $T = 100$.

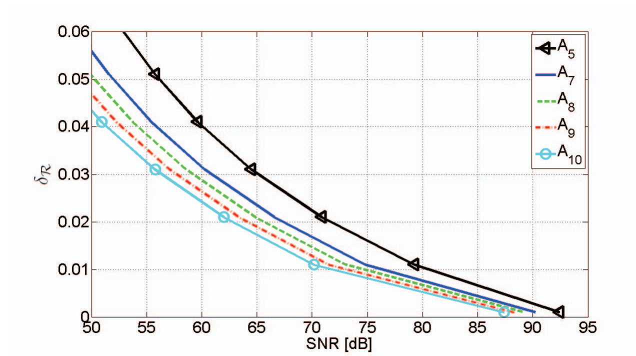

Fig. 2. The LSRL versus the required SNR to resolve two closely targets for $T = 100$, $L = 10$, a transmitter ULA with $N_{\mathcal{T}} = 10$ and for different $A_{N_{\mathcal{R}}}$ of $N_{\mathcal{R}} \in \{5, 7, 8, 9, 10\}$.

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