Recursive Hybrid CRB for Markovian Systems with Time-Variant Measurement Parameters

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***Abstract*—In statistical signal processing, hybrid parameter estimation refers to the case where the parameters vector to estimate contains both deterministic and random parameters. Lately computationally tractable hybrid Cramér-Rao lower bounds for discrete-time Markovian dynamic systems depending on unknown time invariant deterministic parameters has been released. However in many applications (radar, sonar, telecoms, ...) the unknown deterministic parameters of the measurement model are time variant which prevents from using the aforementioned bounds. It is therefore the aim of this communication to tackle this issue by introducing new computationally tractable hybrid Cramér-Rao lower bounds.**

I. Introduction

In the Bayesian estimation framework, discrete-time Markovian dynamic systems (MDS) arises in various applications such as adaptive control, analysis and prediction of nonstationary time series [1], or signal source tracking (radar, sonar, telecoms) [2]. As is well known, the optimal estimator for this problem cannot be built in general, and it is necessary to turn to one of the large number of existing suboptimal filtering techniques [1][2]. Assessing the achievable performance may be difficult, and we have to resort to simulations and comparing proximity to Bayesian lower bounds corresponding to optimum performance [2]. Actually, most discrete-time MDS incorporate some deterministic parameters which can be either known or unknown according to the experimental conditions [2]. Even when the deterministic parameters are known, some of the true values may originate from a prior calibration process which accuracy impacts on the optimum performance of random parameter estimates. In both cases, there is a need for computationally tractable hybrid lower bounds for discrete-time MDS depending on unknown deterministic parameters.

The first hybrid lower bound, the so-called hybrid Cramér-Rao bound (HCRB), has been introduced in [3], extended in [4] to the case where the prior probability density function (pdf) of the random parameters depends on deterministic parameters, and its asymptotic tightness has been further analyzed in [5]. This characterization of hybrid estimation has been generalized by Reuven and Messer [6] who introduced the first "large-error" hybrid bound, the so-called hybrid Barankin Bound (HBB), in order to handle the threshold phenomena and of which one limiting form yields the HCRB. This seminal work [6] has been lately extended to new "large-error" hybrid bounds [7][8][9] in order to improve the estimation of the transition region where the threshold phenomena occurs. Unfortunately, the computational cost of hybrid "large-error" bounds is prohibitive in most applications when the number of unknown parameters increases. All these works have shown that, like the deterministic CRB and Bayesian CRB (BCRB), the HCRB is valid in the asymptotic region only, i.e., when signal to noise ratio is high or the number of observations (measurements) is large.

Therefore, provided that one keeps in mind its limitations, the HCRB is a lower bound of great interest for system analysis and design in the asymptotic region. For that purpose, computationally tractable HCRB for discrete-time MDS depending on unknown time invariant deterministic parameters has been released only lately in [10] and extended in [11]. However in many applications (radar, sonar, telecoms, ...) the unknown deterministic parameters of the measurement model are time variant which prevents from using the bounds derived in [11]. It is therefore the aim of this communication to tackle this issue by introducing new computationally tractable HCRB.

II. Background on HCRB for MDS with unknown deterministic parameters

In hybrid parameter estimation one wishes to estimate an unknown hybrid parameter vector $(\mathbf{x};\boldsymbol{\theta})$ from a random observation vector $\mathbf{y} \in \mathbb{R}^{N'}$, where for L column vectors $\mathbf{a}_l$, $(\mathbf{a}_1;\mathbf{a}_2;\ldots;\mathbf{a}_L) \triangleq (\mathbf{a}_1^T,\mathbf{a}_2^T,\ldots,\mathbf{a}_L^T)^T$ denotes the vertical concatenation. Some prior knowledge is available on random parameter $\mathbf{x} \in \mathbb{R}^{P'}$ that is incorporated by an *a priori* pdf $p(\mathbf{x})$ which support is a subset Π_r of $\mathbb{R}^{P'}$. No such knowledge is available on $\boldsymbol{\theta} \in \Pi_d \subset \mathbb{R}^{D'}$ and thus it is considered deterministic. In the general case, $p(\mathbf{x})$ may depend on the unknown parameter $\boldsymbol{\theta}$, and it is denoted $p(\mathbf{x}|\boldsymbol{\theta})$. The conditional pdf of $\mathbf{y}$ given $\mathbf{x}$ parameterized by $\boldsymbol{\theta}$ is $p(\mathbf{y}|\mathbf{x},\boldsymbol{\theta})$ and their joint pdf parameterized by $\boldsymbol{\theta}$ is given by $p(\mathbf{y},\mathbf{x}|\boldsymbol{\theta}) = p(\mathbf{y}|\mathbf{x},\boldsymbol{\theta})\, p(\mathbf{x}|\boldsymbol{\theta})$. Then, for any estimators $\widehat{\boldsymbol{\theta}}(\mathbf{y})$

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of $\boldsymbol{\theta}$ and $\widehat{\mathbf{x}}(\mathbf{y})$ of $\mathbf{x}$, the HCRB is given by [3][4][5][11]:

$$E_{\mathbf{y},\mathbf{x}|\boldsymbol{\theta}}\left[\mathbf{e}(\mathbf{y})\,\mathbf{e}(\mathbf{y})^T\right] \succeq \mathbf{HCRB}_{(\mathbf{x};\boldsymbol{\theta})} = \mathbf{J}_{(\mathbf{x};\boldsymbol{\theta})}^{-1}, \quad (1)$$

$$\mathbf{J}_{(\mathbf{x};\boldsymbol{\theta})} = E_{\mathbf{y},\mathbf{x}|\boldsymbol{\theta}}\left[\frac{\partial \ln p(\mathbf{y},\mathbf{x}|\boldsymbol{\theta})}{\partial(\mathbf{x};\boldsymbol{\theta})}\frac{\partial \ln p(\mathbf{y},\mathbf{x}|\boldsymbol{\theta})}{\partial(\mathbf{x};\boldsymbol{\theta})^T}\right], \quad (2)$$

where $\mathbf{e}(\mathbf{y}) = (\widehat{\mathbf{x}}(\mathbf{y}) - \mathbf{x}; \widehat{\boldsymbol{\theta}}(\mathbf{y}) - \boldsymbol{\theta})$, $E_{\mathbf{y},\mathbf{x}|\boldsymbol{\theta}}[\mathbf{g}(\mathbf{y},\mathbf{x})]$ is the statistical expectation of the vector of functions $\mathbf{g}(\ )$ with respect to $\mathbf{y}$ and $\mathbf{x}$ parameterized by $\boldsymbol{\theta}$, and for two matrices, $\mathbf{A} \succeq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is positive semi-definite. The regularity conditions for the hybrid Fisher information matrix (HFIM) $\mathbf{J}_{(\mathbf{x};\boldsymbol{\theta})}$ to be of the usual form (2) are [11]:
(R1): $\Pi_r = \mathbb{R}^{P'}$,
(R2): $E_{\mathbf{y},\mathbf{x}|\boldsymbol{\theta}}\left[\frac{\partial \ln p(\mathbf{y},\mathbf{x}|\boldsymbol{\theta})}{\partial x_{p'}}^2\right], E_{\mathbf{y},\mathbf{x}|\boldsymbol{\theta}}\left[\frac{\partial \ln p(\mathbf{y},\mathbf{x}|\boldsymbol{\theta})}{\partial \theta_{d'}}^2\right] < \infty$.
Moreover, under its usual form (1), the HCRB is a lower bound for the class of estimates satisfying (R3):
$E_{\mathbf{y},\mathbf{x}|\boldsymbol{\theta}}\left[\left(\widehat{\mathbf{x}}(\mathbf{y}) - \mathbf{x}; \widehat{\boldsymbol{\theta}}(\mathbf{y}) - \boldsymbol{\theta}\right)\right] = (\boldsymbol{\mu}; \mathbf{0})$.
Our main concern is the derivation of a computationally tractable HFIM (2) for hybrid discrete-time MDS represented with the state and measurement equations:

$$\mathbf{x}_k = \mathbf{f}_{k-1}(\mathbf{x}_{k-1}, \mathbf{w}_{k-1}, \boldsymbol{\alpha}),\ \mathbf{y}_k = \mathbf{h}_k(\mathbf{x}_k, \mathbf{v}_k, \boldsymbol{\lambda}_k) \quad (3)$$

where $k \geq 1$ is a time index, $\mathbf{x}_k$ is the P-dimensional state vector, $\mathbf{y}_k$ is the N-dimensional measurement vector, $f_k(\ ,\ ,\boldsymbol{\alpha})$ and $\mathbf{h}_k(\ ,\ ,\boldsymbol{\lambda}_k)$ are known parametric vector functions depending on an unknown deterministic parameter vector ($\boldsymbol{\alpha}$ and $\boldsymbol{\lambda}_k$ respectively). The process noise sequence $\{\mathbf{w}_k\}$ and the measurement noise sequence $\{\mathbf{v}_k\}$ are mutually independent white sequences described by known pdfs $p(\mathbf{w}_k|\boldsymbol{\beta})$ and $p(\mathbf{v}_k|\boldsymbol{\mu}_k)$, respectively, depending on an unknown deterministic parameter vector ($\boldsymbol{\beta}$ and $\boldsymbol{\mu}_k$ respectively). The noises are independent of the initial state $\mathbf{x}_0$ described by the known pdf $p(\mathbf{x}_0|\boldsymbol{\alpha})$.
Let adopt the notational convention: $\forall l \leq k, \overline{\boldsymbol{\delta}}_{l:k} = (\boldsymbol{\delta}_l; \ldots; \boldsymbol{\delta}_k)$ where all vectors are of same dimension. At time indexes $1, 2, \ldots, k$ the dependency of the state and measurement on deterministic unknown parameters is as follows:

$$\left|\begin{array}{ll} \mathbf{x}_1 \triangleq \mathbf{x}_1(\boldsymbol{\alpha},\boldsymbol{\beta}) & = \mathbf{f}_0(\mathbf{x}_0, \mathbf{w}_0(\boldsymbol{\beta}), \boldsymbol{\alpha}) \\ \mathbf{y}_1 \triangleq \mathbf{y}_1(\boldsymbol{\lambda}_1, \boldsymbol{\mu}_1, \boldsymbol{\alpha}, \boldsymbol{\beta}) & = \mathbf{h}_1(\mathbf{x}_1(\boldsymbol{\alpha},\boldsymbol{\beta}), \mathbf{v}_1(\boldsymbol{\mu}_1), \boldsymbol{\lambda}_1) \end{array}\right.,$$
$$\left|\begin{array}{ll} \mathbf{x}_2 \triangleq \mathbf{x}_2(\boldsymbol{\alpha},\boldsymbol{\beta}) & = \mathbf{f}_1(\mathbf{x}_1(\boldsymbol{\alpha},\boldsymbol{\beta}), \mathbf{w}_1(\boldsymbol{\beta}), \boldsymbol{\alpha}) \\ \mathbf{y}_2 \triangleq \mathbf{y}_2(\boldsymbol{\lambda}_2, \boldsymbol{\mu}_2, \boldsymbol{\alpha}, \boldsymbol{\beta}) & = \mathbf{h}_2(\mathbf{x}_2(\boldsymbol{\alpha},\boldsymbol{\beta}), \mathbf{v}_2(\boldsymbol{\mu}_2), \boldsymbol{\lambda}_2) \end{array}\right.,$$
$$\vdots$$
$$\left|\begin{array}{ll} \mathbf{x}_k \triangleq \mathbf{x}_k(\boldsymbol{\alpha},\boldsymbol{\beta}) & = \mathbf{f}_{k-1}(\mathbf{x}_{k-1}(\boldsymbol{\alpha},\boldsymbol{\beta}), \mathbf{w}_{k-1}(\boldsymbol{\beta}), \boldsymbol{\alpha}) \\ \mathbf{y}_k \triangleq \mathbf{y}_k(\boldsymbol{\lambda}_k, \boldsymbol{\mu}_k, \boldsymbol{\alpha}, \boldsymbol{\beta}) & = \mathbf{h}_k(\mathbf{x}_k(\boldsymbol{\alpha},\boldsymbol{\beta}), \mathbf{v}_k(\boldsymbol{\mu}_k), \boldsymbol{\lambda}_k) \end{array}\right. \quad (4)$$

Therefore, the dependency of the state and measurement on deterministic unknown parameters on a horizon of k points from the first measurement is as follows:

$$k \geq 1: \left|\begin{array}{lll} \overline{\mathbf{x}}_{0:k} \triangleq \overline{\mathbf{x}}_{0:k}(\boldsymbol{\alpha},\boldsymbol{\beta}) & & \triangleq \overline{\mathbf{x}}_{0:k}(\boldsymbol{\theta}_0) \\ \overline{\mathbf{y}}_{1:k} \triangleq \overline{\mathbf{y}}_{1:k}\left(\overline{\boldsymbol{\lambda}}_{1:k}, \overline{\boldsymbol{\mu}}_{1:k}, \boldsymbol{\alpha}, \boldsymbol{\beta}\right) & & \triangleq \overline{\mathbf{y}}_{1:k}(\boldsymbol{\theta}_k) \end{array}\right., \quad (5)$$

$$\boldsymbol{\theta}_0 = (\boldsymbol{\alpha}; \boldsymbol{\beta}), \quad k \geq 1: \boldsymbol{\theta}_k = \left(\overline{\boldsymbol{\lambda}}_{1:k}; \overline{\boldsymbol{\mu}}_{1:k}; \boldsymbol{\theta}_0\right), \quad (6)$$

where $\boldsymbol{\theta}_k$ is the vector gathering all the unknown deterministic parameters. The state transition and the measurement pdfs depend on unknown deterministic parameters:

$$p(\mathbf{x}_k|\mathbf{x}_{k-1}) \triangleq p(\mathbf{x}_k|\mathbf{x}_{k-1}, \boldsymbol{\theta}_0),\ p(\mathbf{y}_k|\mathbf{x}_k) \triangleq p(\mathbf{y}_k|\mathbf{x}_k, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_k, \boldsymbol{\theta}_0), \quad (7)$$

and we suppose that both $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \boldsymbol{\theta}_0)$ and $p(\mathbf{y}_k|\mathbf{x}_k, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_k, \boldsymbol{\theta}_0)$ are twice differentiable with respect to all their arguments. Since (4) is a MDS:

$$p(\overline{\mathbf{y}}_{1:k}, \overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k) = p(\mathbf{x}_0|\boldsymbol{\alpha})\prod_{l=1}^{k} p(\mathbf{y}_l|\mathbf{x}_l, \boldsymbol{\lambda}_l, \boldsymbol{\mu}_l, \boldsymbol{\theta}_0)\, p(\mathbf{x}_l|\mathbf{x}_{l-1}, \boldsymbol{\theta}_0) \quad (8)$$

From a theoretical point of view, we are primarily interested in the HFIM (2) on $(\mathbf{x};\boldsymbol{\theta}) \triangleq (\mathbf{x}_k;\boldsymbol{\theta}_k)$ associated to the measurement vector $\mathbf{y} \triangleq \overline{\mathbf{y}}_{1:k}$ resulting from the set of k measurements $\mathbf{y}_1,...,\mathbf{y}_k$:

$$\mathbf{J}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)} = E_{\overline{\mathbf{y}}_{1:k},\mathbf{x}_k|\boldsymbol{\theta}_k}\left[\frac{\partial \ln p(\overline{\mathbf{y}}_{1:k}, \mathbf{x}_k|\boldsymbol{\theta}_k)}{\partial(\mathbf{x}_k;\boldsymbol{\theta}_k)}\frac{\partial \ln p(\overline{\mathbf{y}}_{1:k}, \mathbf{x}_k|\boldsymbol{\theta}_k)}{\partial(\mathbf{x}_k;\boldsymbol{\theta}_k)^T}\right]$$

which alternative formula is [3][4][5][6]:

$$\mathbf{J}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)} = E_{\overline{\mathbf{y}}_{1:k},\mathbf{x}_k|\boldsymbol{\theta}_k}\left[\frac{-\partial^2 \ln p(\overline{\mathbf{y}}_{1:k}, \mathbf{x}_k|\boldsymbol{\theta}_k)}{\partial(\mathbf{x}_k;\boldsymbol{\theta}_k)\,\partial(\mathbf{x}_k;\boldsymbol{\theta}_k)^T}\right] \quad (9)$$

Unfortunately the computation of (9) requires the derivation of the marginal pdf $p(\overline{\mathbf{y}}_{1:k}, \mathbf{x}_k|\boldsymbol{\theta}_k)$ from (8) which is generally mathematically intractable [2][5][12]. However an upper bound of (9) can be derived from the HFIM (2) on $(\mathbf{x};\boldsymbol{\theta}_k) \triangleq (\overline{\mathbf{x}}_{0:k};\boldsymbol{\theta}_k)$ associated to the measurement vector $\mathbf{y} \triangleq \overline{\mathbf{y}}_{1:k}$ [11]:

$$\mathbf{J}_{(\overline{\mathbf{x}}_{0:k};\boldsymbol{\theta}_k)} = E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[\frac{-\partial^2 \ln p(\overline{\mathbf{y}}_{1:k}, \overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k)}{\partial(\overline{\mathbf{x}}_{0:k};\boldsymbol{\theta}_k)\,\partial(\overline{\mathbf{x}}_{0:k};\boldsymbol{\theta}_k)^T}\right]. \quad (10)$$

Indeed, if we decompose $\mathbf{J}_{(\overline{\mathbf{x}}_{0:k};\boldsymbol{\theta}_k)}$ as:

$$\mathbf{J}_{(\overline{\mathbf{x}}_{0:k};\boldsymbol{\theta}_k)} = \begin{bmatrix} \mathbf{J}_k^{11} & \mathbf{B}_k \\ \mathbf{B}_k^T & \mathbf{C}_k \end{bmatrix}, \quad \begin{array}{l} \mathbf{J}_k^{11} = E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k)}{\partial\overline{\mathbf{x}}_{0:k-1}\partial\overline{\mathbf{x}}_{0:k-1}^T}\right] \\ \mathbf{B}_k = E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k)}{\partial\overline{\mathbf{x}}_{0:k-1}\partial(\mathbf{x}_k;\boldsymbol{\theta}_k)^T}\right] \\ \mathbf{C}_k = E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\mathbf{x}_k|\boldsymbol{\theta}_k)}{\partial(\mathbf{x}_k;\boldsymbol{\theta}_k)\partial(\mathbf{x}_k;\boldsymbol{\theta}_k)^T}\right] \end{array},$$

then [11]:

$$\widetilde{\mathbf{J}}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)} \succeq \mathbf{J}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)} \ \Leftrightarrow\ \mathbf{HCRB}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)} \succeq \widetilde{\mathbf{J}}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)}^{-1}, \quad (11)$$

$$\widetilde{\mathbf{J}}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)} = \mathbf{C}_k - \mathbf{B}_k^T\left(\mathbf{J}_k^{11}\right)^{-1}\mathbf{B}_k. \quad (12)$$

$\widetilde{\mathbf{J}}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)}^{-1}$ defines a looser (in comparison with $\mathbf{HCRB}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)}$) but general computable hybrid bound for discrete-time MDS pdf (8). However the computation of $\widetilde{\mathbf{J}}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)}^{-1}$ (11) may become rapidly computationally prohibitive and unstable since $\mathbf{J}_k^{11}$ and $\widetilde{\mathbf{J}}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)}$ are matrices with unbounded dimension as the time index k increases (a $(PK \times PK)$ matrix and a $((P + \dim\boldsymbol{\theta}_k) \times (P + \dim\boldsymbol{\theta}_k))$ matrix, respectively). Hence the need of a recursive form involving matrices with bounded dimensions.

III. A NEW RECURSIVE HCRB FOR MDS WITH TIME-VARIANT MEASUREMENT PARAMETERS

From a practical point of view, in estimation problem such (3)(4), at each time index k, the vector of parameters of interest is rather $(\mathbf{x}_k;\boldsymbol{\omega}_k)$, $\boldsymbol{\omega}_k = (\boldsymbol{\lambda}_k;\boldsymbol{\mu}_k;\boldsymbol{\alpha};\boldsymbol{\beta})$, than $(\mathbf{x}_k;\boldsymbol{\theta}_k)$. In that perspective, we can reparameterize $(\overline{\mathbf{x}}_{0:k};\boldsymbol{\theta}_k)$ as:

$$\begin{pmatrix} \overline{\mathbf{z}}_{0:k} \\ \boldsymbol{\theta}_0 \end{pmatrix}, \quad \left| \begin{array}{l} k=0 : \mathbf{z}_0 = \mathbf{x}_0 \\ k \geq 1 : \mathbf{z}_k = (\mathbf{x}_k;\boldsymbol{\lambda}_k;\boldsymbol{\mu}_k) \end{array} \right. , \tag{13}$$

and rearrange $(\mathbf{x}_k;\boldsymbol{\theta}_k)$ as:

$$\mathbf{v}_k = \left(\gamma_{k-1};\mathbf{z}_k;\boldsymbol{\theta}_0\right), \quad \gamma_{k-1} = \left(\overline{\boldsymbol{\lambda}}_{1:k-1};\overline{\boldsymbol{\mu}}_{1:k-1}\right). \tag{14}$$

Then:

$$\mathbf{HCRB}_{\mathbf{v}_k} = \begin{bmatrix} \mathbf{HCRB}_{\mathbf{v}_k}^{\gamma_{k-1}} & \mathbf{HCRB}_{\mathbf{v}_k}^{\gamma_{k-1},(\mathbf{z}_k;\boldsymbol{\theta}_0)} \\ \mathbf{HCRB}_{\mathbf{v}_k}^{(\mathbf{z}_k;\boldsymbol{\theta}_0),\gamma_{k-1}} & \mathbf{HCRB}_{\mathbf{v}_k}^{(\mathbf{z}_k;\boldsymbol{\theta}_0)} \end{bmatrix}$$

and:

$$\boxed{\mathbf{HCRB}_{(\mathbf{x}_k;\boldsymbol{\theta}_k)}^{(\mathbf{x}_k;\boldsymbol{\omega}_k)} = \mathbf{HCRB}_{\mathbf{v}_k}^{(\mathbf{z}_k;\boldsymbol{\theta}_0)} \succeq \widetilde{\mathbf{J}}_{(\mathbf{z}_k;\boldsymbol{\theta}_0)}^{-1}} \tag{15}$$

where $\widetilde{\mathbf{J}}_{(\mathbf{z}_k;\boldsymbol{\theta}_0)}$ can be decomposed into block matrices:

$$\widetilde{\mathbf{J}}_{(\mathbf{z}_k;\boldsymbol{\theta}_0)} = \begin{bmatrix} \widetilde{\mathbf{J}}_k^{\mathbf{z}_k,\mathbf{z}_k} & \widetilde{\mathbf{J}}_k^{\mathbf{z}_k,\boldsymbol{\theta}_0} \\ \widetilde{\mathbf{J}}_k^{\boldsymbol{\theta}_0,\mathbf{z}} & \widetilde{\mathbf{J}}_k^{\boldsymbol{\theta}_0,\boldsymbol{\theta}_0} \end{bmatrix} \tag{16}$$

which obey the recursion:

$$\left| \begin{array}{l} \widetilde{\mathbf{J}}_k^{\mathbf{z}_k,\mathbf{z}_k} = \mathbf{D}_{k-1}^{22} - \left(\mathbf{D}_{k-1}^{12}\right)^T \left(\mathbf{D}_{k-1}^{11} + \widetilde{\mathbf{J}}_{k-1}^{\mathbf{z}_{k-1},\mathbf{z}_{k-1}}\right)^{-1} \mathbf{D}_{k-1}^{12} \\ \widetilde{\mathbf{J}}_k^{\mathbf{z}_k,\boldsymbol{\theta}_0} = \mathbf{D}_{k-1}^{23} - \left(\mathbf{D}_{k-1}^{12}\right)^T \left(\mathbf{D}_{k-1}^{11} + \widetilde{\mathbf{J}}_{k-1}^{\mathbf{z}_{k-1},\mathbf{z}_{k-1}}\right)^{-1} \\ \qquad\qquad \times \left(\mathbf{D}_{k-1}^{13} + \widetilde{\mathbf{J}}_{k-1}^{\mathbf{z}_{k-1},\boldsymbol{\theta}_0}\right) \\ \widetilde{\mathbf{J}}_k^{\boldsymbol{\theta}_0,\boldsymbol{\theta}_0} = \mathbf{D}_{k-1}^{33} + \widetilde{\mathbf{J}}_{k-1}^{\boldsymbol{\theta}_0,\boldsymbol{\theta}_0} - \left(\mathbf{D}_{k-1}^{13} + \widetilde{\mathbf{J}}_{k-1}^{\mathbf{z}_{k-1},\boldsymbol{\theta}_0}\right)^T \\ \qquad \times \left(\mathbf{D}_{k-1}^{11} + \widetilde{\mathbf{J}}_{k-1}^{\mathbf{z}_{k-1},\mathbf{z}_{k-1}}\right)^{-1} \left(\mathbf{D}_{k-1}^{13} + \widetilde{\mathbf{J}}_{k-1}^{\mathbf{z}_{k-1},\boldsymbol{\theta}_0}\right) \end{array} \right. \tag{17}$$

where:

$$\left| \begin{array}{l} \mathbf{D}_{k-1}^{11} = E_{\mathbf{x}_k,\mathbf{x}_{k-1}|\boldsymbol{\theta}_0}\left[-\frac{\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1},\boldsymbol{\theta}_0)}{\partial \mathbf{z}_{k-1}\partial \mathbf{z}_{k-1}^T}\right] \\ \mathbf{D}_{k-1}^{12} = E_{\mathbf{x}_k,\mathbf{x}_{k-1}|\boldsymbol{\theta}_0}\left[-\frac{\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1},\boldsymbol{\theta}_0)}{\partial \mathbf{z}_{k-1}\partial \mathbf{z}_k^T}\right] \\ \mathbf{D}_{k-1}^{22} = E_{\mathbf{y}_k,\mathbf{x}_k|\boldsymbol{\lambda}_k,\boldsymbol{\mu}_k,\boldsymbol{\theta}_0}\left[-\frac{\partial^2 \ln p(\mathbf{y}_k|\mathbf{x}_k,\boldsymbol{\lambda}_k,\boldsymbol{\mu}_k,\boldsymbol{\theta}_0)}{\partial \mathbf{z}_k\partial \mathbf{z}_k^T}\right] \\ \qquad\qquad + E_{\mathbf{x}_k,\mathbf{x}_{k-1}|\boldsymbol{\theta}_0}\left[-\frac{\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1},\boldsymbol{\theta}_0)}{\partial \mathbf{z}_k\partial \mathbf{z}_k^T}\right] \end{array} \right. \tag{18}$$

$$\left| \begin{array}{l} \mathbf{D}_{k-1}^{13} = E_{\mathbf{x}_k,\mathbf{x}_{k-1}|\boldsymbol{\theta}_0}\left[-\frac{\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1},\boldsymbol{\theta}_0)}{\partial \mathbf{z}_{k-1}\partial \boldsymbol{\theta}_0^T}\right] \\ \mathbf{D}_{k-1}^{23} = E_{\mathbf{y}_k,\mathbf{x}_k|\boldsymbol{\lambda}_k,\boldsymbol{\mu}_k,\boldsymbol{\theta}_0}\left[-\frac{\partial^2 \ln p(\mathbf{y}_k|\mathbf{x}_k,\boldsymbol{\lambda}_k,\boldsymbol{\mu}_k,\boldsymbol{\theta}_0)}{\partial \mathbf{z}_k\partial \boldsymbol{\theta}_0^T}\right] \\ \qquad\qquad + E_{\mathbf{x}_k,\mathbf{x}_{k-1}|\boldsymbol{\theta}_0}\left[-\frac{\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1},\boldsymbol{\theta}_0)}{\partial \mathbf{z}_k\partial \boldsymbol{\theta}_0^T}\right] \\ \mathbf{D}_{k-1}^{33} = E_{\mathbf{y}_k,\mathbf{x}_k|\boldsymbol{\lambda}_k,\boldsymbol{\mu}_k,\boldsymbol{\theta}_0}\left[-\frac{\partial^2 \ln p(\mathbf{y}_k|\mathbf{x}_k,\boldsymbol{\lambda}_k,\boldsymbol{\mu}_k,\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}_0\partial \boldsymbol{\theta}_0^T}\right] \\ \qquad\qquad + E_{\mathbf{x}_k,\mathbf{x}_{k-1}|\boldsymbol{\theta}_0}\left[-\frac{\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1},\boldsymbol{\theta}_0)}{\partial \boldsymbol{\theta}_0\partial \boldsymbol{\theta}_0^T}\right] \end{array} \right. \tag{19}$$

$$\widetilde{\mathbf{J}}_{(\mathbf{z}_0;\boldsymbol{\theta}_0)} = \mathbf{J}_{(\mathbf{z}_0;\boldsymbol{\theta}_0)} = E_{\mathbf{x}_0|\boldsymbol{\theta}_0}\left[\frac{-\partial^2 \ln p\left(\mathbf{x}_0|\boldsymbol{\alpha}\right)}{\partial\left(\mathbf{x}_0;\boldsymbol{\theta}_0\right)\partial\left(\mathbf{x}_0;\boldsymbol{\theta}_0\right)^T}\right] \tag{20}$$

Proof:
The first step consists in reformulating (11)(12) after the rearrangement (14) leading to:

$$\widetilde{\mathbf{J}}_{\mathbf{v}_k} \succeq \mathbf{J}_{\mathbf{v}_k}, \quad \widetilde{\mathbf{J}}_{\mathbf{v}_k} = \underline{\mathbf{C}}_k - \underline{\mathbf{B}}_k^T\left(\mathbf{J}_k^{11}\right)^{-1}\underline{\mathbf{B}}_k, \tag{21}$$

$$\begin{aligned} \mathbf{J}_k^{11} &= E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k)}{\partial \overline{\mathbf{x}}_{0:k-1}\partial \overline{\mathbf{x}}_{0:k-1}^T}\right] \\ \underline{\mathbf{B}}_k &= E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k)}{\partial \overline{\mathbf{x}}_{0:k-1}\partial \mathbf{v}_k^T}\right] \\ \underline{\mathbf{C}}_k &= E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\mathbf{x}_k|\boldsymbol{\theta}_k)}{\partial \mathbf{v}_k\partial \mathbf{v}_k^T}\right] \end{aligned}$$

Let $\mathbf{u}_k = \left(\overline{\mathbf{x}}_{0:k-1};\gamma_{k-1};\mathbf{z}_k;\boldsymbol{\theta}_0\right) \triangleq \left(\overline{\mathbf{x}}_{0:k-1};\mathbf{v}_k\right)$, that is $\left(\overline{\mathbf{x}}_{0:k};\boldsymbol{\theta}_k\right)$ after rearrangement, then:

$$\mathbf{J}_{\mathbf{u}_k} = E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p\left(\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k\right)}{\partial \mathbf{u}_k\partial \mathbf{u}_k^T}\right] = \begin{bmatrix} \mathbf{J}_k^{11} & \underline{\mathbf{B}}_k \\ \underline{\mathbf{B}}_k^T & \underline{\mathbf{C}}_k \end{bmatrix}.$$

Therefore, using block matrix inversion:

$$\mathbf{HCRB}_{\mathbf{u}_k} = \mathbf{J}_{\mathbf{u}_k}^{-1} = \begin{bmatrix} \mathbf{HCRB}_{\mathbf{u}_k}^{\overline{\mathbf{x}}_{0:k-1}} & \mathbf{HCRB}_{\mathbf{u}_k}^{\overline{\mathbf{x}}_{0:k-1},\mathbf{v}_k} \\ \mathbf{HCRB}_{\mathbf{u}_k}^{\mathbf{v}_k,\overline{\mathbf{x}}_{0:k-1}} & \mathbf{HCRB}_{\mathbf{u}_k}^{\mathbf{v}_k} \end{bmatrix},$$

$$\mathbf{HCRB}_{\mathbf{u}_k}^{\mathbf{v}_k} = \left(\underline{\mathbf{C}}_k - \underline{\mathbf{B}}_k^T\left(\mathbf{J}_k^{11}\right)^{-1}\underline{\mathbf{B}}_k\right)^{-1} = \widetilde{\mathbf{J}}_{\mathbf{v}_k}^{-1}.$$

Additionally, according to (13)(14):

$$\overline{\mathbf{z}}_{0:k-1} = \mathbf{P}_{0:k-1}\begin{pmatrix} \overline{\mathbf{x}}_{0:k-1} \\ \gamma_{k-1} \end{pmatrix} \Rightarrow \begin{pmatrix} \overline{\mathbf{z}}_{0:k-1} \\ \begin{pmatrix} \mathbf{z}_k \\ \boldsymbol{\theta}_0 \end{pmatrix} \end{pmatrix} = \begin{bmatrix} \mathbf{P}_{0:k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{u}_k$$

where $\mathbf{P}_{0:k}$ is a permutation matrix; therefore:

$$\mathbf{HCRB}_{\mathbf{u}_k} = \begin{bmatrix} \mathbf{P}_{0:k-1}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \mathbf{HCRB}_{(\overline{\mathbf{z}}_{0:k};\boldsymbol{\theta}_0)} \begin{bmatrix} \mathbf{P}_{0:k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

leading to the key identity:

$$\mathbf{HCRB}_{\mathbf{u}_k}^{(\mathbf{z}_k;\boldsymbol{\theta}_0)} = \mathbf{HCRB}_{(\overline{\mathbf{z}}_{0:k};\boldsymbol{\theta}_0)}^{(\mathbf{z}_k;\boldsymbol{\theta}_0)}. \tag{22}$$

Indeed, (21) yields:

$$\mathbf{J}_{\mathbf{v}_k}^{-1} \succeq \widetilde{\mathbf{J}}_{\mathbf{v}_k}^{-1} \Leftrightarrow \mathbf{HCRB}_{\mathbf{v}_k} \succeq \mathbf{HCRB}_{\mathbf{u}_k}^{\mathbf{v}_k},$$

that is, in particular:

$$\mathbf{HCRB}_{\mathbf{v}_k}^{(\mathbf{z}_k;\boldsymbol{\theta}_0)} \succeq \mathbf{HCRB}_{\mathbf{u}_k}^{(\mathbf{z}_k;\boldsymbol{\theta}_0)} = \mathbf{HCRB}_{(\overline{\mathbf{z}}_{0:k};\boldsymbol{\theta}_0)}^{(\mathbf{z}_k;\boldsymbol{\theta}_0)}. \tag{23}$$

The second step consists in showing that

$$\widetilde{\mathbf{J}}_{(\mathbf{z}_k;\boldsymbol{\theta}_0)} = \left(\mathbf{HCRB}_{(\overline{\mathbf{z}}_{0:k};\boldsymbol{\theta}_0)}^{(\mathbf{z}_k;\boldsymbol{\theta}_0)}\right)^{-1} = \begin{bmatrix} \widetilde{\mathbf{J}}_k^{\mathbf{z}_k,\mathbf{z}_k} & \widetilde{\mathbf{J}}_k^{\mathbf{z}_k,\boldsymbol{\theta}_0} \\ \widetilde{\mathbf{J}}_k^{\boldsymbol{\theta}_0,\mathbf{z}_k} & \widetilde{\mathbf{J}}_k^{\boldsymbol{\theta}_0,\boldsymbol{\theta}_0} \end{bmatrix}$$

obey the recursion defined by (17-20). First $\mathbf{J}_{(\overline{\mathbf{z}}_{0:k};\boldsymbol{\theta}_0)}$ can be broken down as:

$$\mathbf{J}_{(\overline{\mathbf{z}}_{0:k};\boldsymbol{\theta}_0)} = \begin{bmatrix} \mathbf{J}_k^{11} & \mathbf{J}_k^{12} & \mathbf{J}_k^{13} \\ \left(\mathbf{J}_k^{12}\right)^T & \mathbf{J}_k^{22} & \mathbf{J}_k^{23} \\ \left(\mathbf{J}_k^{13}\right)^T & \left(\mathbf{J}_k^{23}\right)^T & \mathbf{J}_k^{33} \end{bmatrix},$$

$$\begin{aligned} \mathbf{J}_k^{11} &= E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k)}{\partial \overline{\mathbf{z}}_{0:k-1}\partial \overline{\mathbf{z}}_{0:k-1}^T}\right], \\ \mathbf{J}_k^{12} &= E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k)}{\partial \overline{\mathbf{z}}_{0:k-1}\partial \mathbf{z}_k^T}\right], \\ \mathbf{J}_k^{13} &= E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k)}{\partial \overline{\mathbf{z}}_{0:k-1}\partial \boldsymbol{\theta}_0^T}\right], \\ \mathbf{J}_k^{22} &= E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k)}{\partial \mathbf{z}_k\partial \mathbf{z}_k^T}\right], \\ \mathbf{J}_k^{23} &= E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k)}{\partial \mathbf{z}_k\partial \boldsymbol{\theta}_0^T}\right], \\ \mathbf{J}_k^{33} &= E_{\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k}\left[-\frac{\partial^2 \ln p(\overline{\mathbf{y}}_{1:k},\overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k)}{\partial \boldsymbol{\theta}_0^T\partial \boldsymbol{\theta}_0^T}\right]. \end{aligned}$$

Therefore, using block matrix inversion:

$$\widetilde{\mathbf{J}}_{(\mathbf{z}_k;\boldsymbol{\theta}_0)} = \begin{bmatrix} \mathbf{J}_k^{22} & \mathbf{J}_k^{23} \\ \mathbf{J}_k^{32} & \mathbf{J}_k^{33} \end{bmatrix} - \begin{bmatrix} \mathbf{J}_k^{21} \\ \mathbf{J}_k^{31} \end{bmatrix} \left[\mathbf{J}_k^{11}\right]^{-1} \begin{bmatrix} \mathbf{J}_k^{12} & \mathbf{J}_k^{13} \end{bmatrix}$$

that is:

$$\left| \begin{array}{l} \widetilde{\mathbf{J}}_k^{\mathbf{z}_k,\mathbf{z}_k} = \mathbf{J}_k^{22} - \mathbf{J}_k^{21}\left(\mathbf{J}_{k-1}^{11}\right)^{-1}\mathbf{J}_k^{12}, \\ \widetilde{\mathbf{J}}_k^{\boldsymbol{\theta}_0,\mathbf{z}} = \mathbf{J}_k^{23} - \mathbf{J}_k^{21}\left(\mathbf{J}_k^{11}\right)^{-1}\mathbf{J}_k^{13}, \\ \widetilde{\mathbf{J}}_k^{\boldsymbol{\theta}_0,\boldsymbol{\theta}_0} = \mathbf{J}_k^{33} - \mathbf{J}_k^{31}\left(\mathbf{J}_k^{11}\right)^{-1}\mathbf{J}_k^{13}. \end{array} \right. \tag{24}$$

Moreover, for MDS, (8) leads to:

$$p\left(\overline{\mathbf{y}}_{1:k}, \overline{\mathbf{x}}_{0:k}|\boldsymbol{\theta}_k\right) = p\left(\mathbf{y}_k|\mathbf{x}_k, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_k, \boldsymbol{\theta}_0\right) p\left(\mathbf{x}_k|\mathbf{x}_{k-1}, \boldsymbol{\theta}_0\right) \times p\left(\overline{\mathbf{y}}_{1:k-1}, \overline{\mathbf{x}}_{0:k-1}|\boldsymbol{\theta}_{k-1}\right)$$

yielding:

$$\mathbf{J}_k^{11} = \begin{bmatrix} \mathbf{J}_{k-1}^{11} & \mathbf{J}_{k-1}^{12} \\ \mathbf{J}_{k-1}^{21} & \mathbf{J}_{k-1}^{22} + \mathbf{D}_{k-1}^{11} \end{bmatrix}, \quad \mathbf{J}_k^{12} = \begin{bmatrix} \mathbf{0} \\ \mathbf{D}_{k-1}^{12} \end{bmatrix},$$
$$\mathbf{J}_k^{22} = \mathbf{D}_{k-1}^{22}, \quad \mathbf{J}_k^{13} = \begin{bmatrix} \mathbf{J}_{k-1}^{13} \\ \mathbf{J}_{k-1}^{23} + \mathbf{D}_{k-1}^{13} \end{bmatrix},$$
$$\mathbf{J}_k^{23} = \mathbf{D}_{k-1}^{23}, \quad \mathbf{J}_k^{33} = \mathbf{J}_{k-1}^{33} + \mathbf{D}_{k-1}^{33},$$

where $\mathbf{D}_{k-1}^{ij}$ are given by (18-19). Last, using once again block matrix inversion:

$$\left(\mathbf{J}_k^{11}\right)^{-1} = \begin{bmatrix} \boldsymbol{\Phi}_{k-1} & -\boldsymbol{\Gamma}_{k-1}\boldsymbol{\Delta}_{k-1}^{-1} \\ -\boldsymbol{\Delta}_{k-1}^{-1}\boldsymbol{\Gamma}_{k-1}^T & \boldsymbol{\Delta}_{k-1}^{-1} \end{bmatrix},$$
$$\boldsymbol{\Delta}_{k-1} = \mathbf{D}_{k-1}^{11} + \mathbf{J}_{k-1}^{22} - \mathbf{J}_{k-1}^{21}\left(\mathbf{J}_{k-1}^{11}\right)^{-1}\mathbf{J}_{k-1}^{12},$$
$$\boldsymbol{\Gamma}_{k-1} = \left(\mathbf{J}_{k-1}^{11}\right)^{-1}\mathbf{J}_{k-1}^{12},$$
$$\boldsymbol{\Phi}_{k-1} = \left(\mathbf{J}_{k-1}^{11}\right)^{-1} + \boldsymbol{\Gamma}_{k-1}\boldsymbol{\Delta}_{k-1}^{-1}\boldsymbol{\Gamma}_{k-1}^T.$$

Then noting that $\boldsymbol{\Delta}_{k-1} = \mathbf{D}_{k-1}^{11} + \widetilde{\mathbf{J}}_{k-1}^{\mathbf{z}_{k-1},\mathbf{z}_{k-1}}$, a few additional lines of calculus allows to show that equivalent forms of $\widetilde{\mathbf{J}}_k^{\mathbf{z}_k,\mathbf{z}_k}$, $\widetilde{\mathbf{J}}_k^{\boldsymbol{\theta}_0,\boldsymbol{\theta}_0}$and $\widetilde{\mathbf{J}}_k^{\boldsymbol{\theta}_0,\mathbf{z}}$ in (24) are given by (17).

IV. Further considerations

First, some special cases of interest can be easily derived by updating the definitions of $\boldsymbol{\theta}_k$, $p\left(\mathbf{x}_k|\mathbf{x}_{k-1}\right)$, $p\left(\mathbf{y}_k|\mathbf{x}_k\right)$ and $p\left(\mathbf{x}_0|\boldsymbol{\alpha}\right)$ accordingly.

A first case of interest is $\boldsymbol{\theta}_k = \boldsymbol{\theta}_0$, i.e. the measurements do not depend on deterministic parameters, then $\mathbf{z}_k = \mathbf{x}_k$, $p\left(\mathbf{x}_k|\mathbf{x}_{k-1}\right) \triangleq p\left(\mathbf{x}_k|\mathbf{x}_{k-1}, \boldsymbol{\theta}_0\right)$, $p\left(\mathbf{y}_k|\mathbf{x}_k\right) \triangleq p\left(\mathbf{y}_k|\mathbf{x}_k, \boldsymbol{\theta}_0\right)$, and (15-16) reduce to:

$$\mathbf{HCRB}_{(\mathbf{x}_k;\boldsymbol{\theta}_0)} \succeq \widetilde{\mathbf{J}}_{(\mathbf{x}_k;\boldsymbol{\theta}_0)}^{-1}, \ \widetilde{\mathbf{J}}_{(\mathbf{x}_k;\boldsymbol{\theta}_0)} = \begin{bmatrix} \widetilde{\mathbf{J}}_k^{\mathbf{x}_k,\mathbf{x}_k} & \widetilde{\mathbf{J}}_k^{\mathbf{x}_k,\boldsymbol{\theta}_0} \\ \widetilde{\mathbf{J}}_k^{\boldsymbol{\theta}_0,\mathbf{x}_k} & \widetilde{\mathbf{J}}_k^{\boldsymbol{\theta}_0,\boldsymbol{\theta}_0} \end{bmatrix},$$

which are [11, (7)(10)] where $\boldsymbol{\theta} \triangleq (\boldsymbol{\alpha};\boldsymbol{\beta})$, and (17-20) reduce to [11, (11-14)].

A second case of interest is $\boldsymbol{\theta}_k = \varnothing$, i.e. there is no unknown deterministic parameter, then $\mathbf{z}_k = \mathbf{x}_k$, $p\left(\mathbf{x}_k|\mathbf{x}_{k-1}\right) \triangleq p\left(\mathbf{x}_k|\mathbf{x}_{k-1}\right)$, $p\left(\mathbf{y}_k|\mathbf{x}_k\right) \triangleq p\left(\mathbf{y}_k|\mathbf{x}\right)$, and (15-16) and (17-20) reduce to:

$$\mathbf{HCRB}_{\mathbf{x}_k} \succeq \widetilde{\mathbf{J}}_{\mathbf{x}_k}^{-1}, \quad \widetilde{\mathbf{J}}_{\mathbf{x}_k} = \widetilde{\mathbf{J}}_k^{\mathbf{x}_k,\mathbf{x}_k},$$
$$\widetilde{\mathbf{J}}_k^{\mathbf{x}_k,\mathbf{x}_k} = \mathbf{D}_{k-1}^{22} - \left(\mathbf{D}_{k-1}^{12}\right)^T\left(\mathbf{D}_{k-1}^{11} + \widetilde{\mathbf{J}}_{k-1}^{\mathbf{x}_{k-1},\mathbf{x}_{k-1}}\right)^{-1}\mathbf{D}_{k-1}^{12},$$
$$\mathbf{D}_{k-1}^{11} = E_{\mathbf{x}_k,\mathbf{x}_{k-1}}\left[\frac{-\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1})}{\partial \mathbf{x}_{k-1}\partial \mathbf{x}_{k-1}^T}\right],$$
$$\mathbf{D}_{k-1}^{12} = E_{\mathbf{x}_k,\mathbf{x}_{k-1}}\left[\frac{-\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1})}{\partial \mathbf{x}_{k-1}\partial \mathbf{x}_k^T}\right],$$
$$\mathbf{D}_{k-1}^{22} = E_{\mathbf{y}_k,\mathbf{x}_k}\left[\frac{-\partial^2 \ln p(\mathbf{y}_k|\mathbf{x}_k)}{\partial \mathbf{x}_k\partial \mathbf{x}_k^T}\right] + E_{\mathbf{x}_k,\mathbf{x}_{k-1}}\left[\frac{-\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1})}{\partial \mathbf{x}_k\partial \mathbf{x}_k^T}\right]$$

and $\widetilde{\mathbf{J}}_{(\mathbf{z}_0;\boldsymbol{\theta}_0)} \triangleq \mathbf{J}_{\mathbf{x}_0}$, which are (21-25) in [12].

A third case of interest is $\boldsymbol{\theta}_0 = \varnothing$, i.e. the state does not depend on deterministic parameters, then $p\left(\mathbf{x}_k|\mathbf{x}_{k-1}\right) \triangleq p\left(\mathbf{x}_k|\mathbf{x}_{k-1}\right)$, $p\left(\mathbf{y}_k|\mathbf{x}_k\right) \triangleq p\left(\mathbf{y}_k|\mathbf{z}_k\right)$, and (15-16) and (17-20) reduce to:

$$\mathbf{HCRB}_{\mathbf{z}_k} \succeq \widetilde{\mathbf{J}}_{\mathbf{z}_k}^{-1}, \quad \widetilde{\mathbf{J}}_{\mathbf{z}_k} = \widetilde{\mathbf{J}}_k^{\mathbf{z}_k,\mathbf{z}_k},$$
$$\widetilde{\mathbf{J}}_k^{\mathbf{z}_k,\mathbf{z}_k} = \mathbf{D}_{k-1}^{22} - \left(\mathbf{D}_{k-1}^{12}\right)^T\left(\mathbf{D}_{k-1}^{11} + \widetilde{\mathbf{J}}_{k-1}^{\mathbf{z}_{k-1},\mathbf{z}_{k-1}}\right)^{-1}\mathbf{D}_{k-1}^{12},$$
$$\mathbf{D}_{k-1}^{11} = E_{\mathbf{x}_k,\mathbf{x}_{k-1}}\left[\frac{-\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1})}{\partial \mathbf{z}_{k-1}\partial \mathbf{z}_{k-1}^T}\right],$$
$$\mathbf{D}_{k-1}^{12} = E_{\mathbf{x}_k,\mathbf{x}_{k-1}}\left[\frac{-\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1})}{\partial \mathbf{z}_{k-1}\partial \mathbf{z}_k^T}\right],$$
$$\mathbf{D}_{k-1}^{22} = E_{\mathbf{y}_k,\mathbf{x}_k|\boldsymbol{\lambda}_k,\boldsymbol{\mu}_k}\left[\frac{-\partial^2 \ln p(\mathbf{y}_k|\mathbf{z}_k)}{\partial \mathbf{z}_k\partial \mathbf{z}_k^T}\right] + E_{\mathbf{y}_k,\mathbf{x}_k|\boldsymbol{\lambda}_k,\boldsymbol{\mu}_k}\left[\frac{-\partial^2 \ln p(\mathbf{x}_k|\mathbf{x}_{k-1})}{\partial \mathbf{z}_k\partial \mathbf{z}_k^T}\right]$$

and $\widetilde{\mathbf{J}}_{\mathbf{z}_0} \triangleq \mathbf{J}_{\mathbf{x}_0}$ which is a generalization of [12].

Second, it is worth noting that, at each time index k, there is no particular requirements on the dimension of $\boldsymbol{\lambda}_k$ or $\boldsymbol{\mu}_k$; in particular it is possible to have $\boldsymbol{\lambda}_l = \varnothing$ and/or $\boldsymbol{\mu}_l = \varnothing$ for a subset of time index $l \in \{l_1, l_2, \ldots, l_J \leq k\}$, what means that the new HCRB (15) allows to take into account a mixture of measurements where the deterministic parameters are either known or unknown (which is not possible with the HCRB derived in [11]).

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