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Statistical Resolution Limit of the Uniform Linear Cocentered Orthogonal Loop and Dipole Array

Mohammed Nabil El Korso, Rémy Boyer, Alexandre Renaux, and Sylvie Marcos

***Abstract*—Among the family of polarization sensitive arrays, we can find the so-called cocentered orthogonal loop and dipole uniform linear array (COLD-ULA). The COLD-ULA exhibits some interesting properties, e.g., the insensibility of the polarization vector with respect to the source localization in the plan of the array. In this correspondence, we derive the statistical resolution limit (SRL) characterizing the minimal separation, in terms of direction-of-arrivals, to resolve two closely spaced known polarized sources impinging on a COLD-ULA. Toward this end, nonmatrix closed form expressions of the deterministic Cramér–Rao bound (CRB) are derived and thus, the SRL is deduced. A comparison between the SRL of the COLD-ULA and the classical ULA are given. Particularly, it is shown that, in the case of orthogonal known signal sources, the SRL of the COLD-ULA is equal to the SRL of the ULA, meaning that it is not a function of polarization parameters. Furthermore, due to the derived SRL, it is shown that, under some general conditions, the SRL of the COLD-ULA is smaller than the one of the ULA.**

***Index Terms*—Cocentered orthogonal loop and dipole (COLD) array, polarized sources localization, statistical resolution limit.**

I. Introduction

Polarized sources localization by an array of sensors is an important topic with a large number of applications especially in wireless communication and seismology [1]. Particularly, the context of polarized sources has been investigated in the literature and several algorithms, to estimate the localization and polarization parameters, have been proposed [1]–[4]. Among the different types of arrays, the crossed-dipole array (constituted by several couple of dipoles) is sensitive to the source's polarization and thus, is adequate to this context. In particular, the cocentered orthogonal loop and dipole uniform linear array (COLD-ULA) exhibits some interesting properties [5], [6], as for instance, the insensibility of the polarization vector with respect to the source localization in the plan of the array or, the constant norm of the polarization vector. Note that these properties are not shared by the standard crossed-dipole array [5]. The optimal performance in terms of mean square error by way of the Cramér–Rao bound (CRB) for the COLD-ULA array has already been investigated in [5], [6]. In [5], matrix expressions of the CRB was given, whereas, in [6] the asymptotic (in terms of sensors) CRB was derived. However, to the best of our knowledge, no works has been done on the resolvability of closely polarized sources.

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A common tool to characterize the resolvability between two signals is the so-called statistical resolution limit (SRL). The SRL [7]–[19], defined *as the minimal separation between two signals in terms of the parameter of interest*, is a challenging problem and an essential tool to quantify estimator performance.

One can find in the literature three main approaches to characterize the SRL:

i) The first is based on the concept of mean null spectrum and is relevant to a specific high-resolution algorithm [7], [8].
ii) The second approach is based on a hypothesis test using the generalized likelihood ratio test (GLRT) [9]–[11] or the Bayesian approach [12].
iii) The third method is based on the estimation accuracy concept [13]–[18].

In this context, one can distinguish two main criteria. The first one was introduced by Lee in [13] and states that *two signals are resolvable, w.r.t. the parameter of interest ω_1 and ω_2, if the maximum standard deviation, of ω_1 and ω_2, is less than half the difference between ω_1 and ω_2*. However, one can note that the Lee criterion ignores the coupling between the parameters of interest [19]. To take into account this effect, Smith [16], proposed the second following criterion based on the CRB: *two signals are resolvable if the separation between ω_1 and ω_2, is less than the standard deviation of the separation estimation*. Consequently, the SRL in the Smith sense is defined as the separation between the parameters of interest that is equal to the standard deviation of the separation.

To the best of our knowledge, all the works related to the resolvability of closely spaced sources concern the case of non-polarized sources [7]–[9], [11]–[18], and no studies/results are available concerning the case of polarized sources. The goal of this correspondence is to fill this lack.

Since the mean null spectrum approach is relevant to a specific high-resolution algorithm, in this correspondence we focus mainly on the SRL derivation for known polarized sources in the Smith sense. Furthermore, since it exists a relationship between the SRL based on the Smith criterion and the SRL based on a hypothesis test [11] in the asymptotic case, the SRL based on a hypothesis test is deduced and compared to the derived SRL based on the Smith criterion.

Consequently, in this correspondence, we derive and analyze the minimum direction-of-arrivals (DOA) separation between two known polarized sources that allows a correct sources resolvability for the COLD-ULA in the Smith sense. As a by product, a closed-form expression of the true (non-asymptotic) deterministic CRB is given (which is not available in the literature). Finally, the SRL using an ULA is derived and compared to the SRL using a COLD-ULA. It is shown that, in the case of orthogonal known signal sources, the SRL is not a function of polarization parameters (i.e., the SRL of the COLD-ULA is equal to the SRL of the ULA). Furthermore, in the case of non-orthogonal known signal sources and under some general conditions, the SRL of the COLD-ULA is shown to be smaller than the one of the ULA.

II. Model Setup

Consider a COLD-ULA made from L COLD sensors (a COLD sensor is formed by a loop and a dipole [5]) with interelement spacing d that receives a signal emitted by M radiating far-field and narrowband sources. Assuming that the array and the incident signals are coplanar [5], i.e., the elevation is fixed to $\pi/2$, the observed signal model on the ℓ^{th} COLD sensor at the t^{th} snapshot is given by[1] [2], [5] $\boldsymbol{x}_\ell(t) = [x_{\text{loop}}(t)\ x_{\text{dipole}}(t)]^T = \sum_{m=1}^{M} \alpha_m(t)\mathbf{u}_m e^{i\ell\omega_m} + \boldsymbol{v}_\ell(t)$, where $\ell = 0 \dots L-1$ and $t = 1 \dots N$, in which N is the number of snapshots. $\omega_m = (2\pi/\lambda)d\sin(\theta_m)$ is the spatial phase factor in which θ_m and λ are the azimuth of the m^{th} source and the wavelength, respectively. The time-varying source is modelled by[2] $\alpha_m(t) = a_m e^{i(2\pi f_0 t + \phi_m(t))}$ in which a_m is the non-zero real amplitude, $\phi_m(t)$ is the time-varying modulating phase and f_0 denotes the carrier frequency of the incident wave. The additive noise is denoted by $\boldsymbol{v}_\ell(t) = [v_{\text{loop}}(t)\ v_{\text{dipole}}(t)]^T$. The polarization state vector $\mathbf{u}_m$ is given by

$$\mathbf{u}_m = \begin{bmatrix} \frac{2i\pi A_{sl}}{\lambda}\cos(\rho_m) \\ -L_{sd}\sin(\rho_m)e^{i\psi_m} \end{bmatrix}$$

where $\rho_m \in [0, \pi/2]$ and $\psi_m \in [-\pi, \pi]$ are the polarization state parameters. From a modelling point of view, each dipole in the array is assumed to be a short dipole (w.r.t. the distance d) with the same length L_{sd} and each loop is assumed to be a short loop (w.r.t. the distance d) with the same area A_{sl}. Under these assumptions, the total output vector received by the COLD-ULA for the t^{th} snapshot can be written as follows:

$$\begin{aligned} \boldsymbol{y}(t) &= \begin{bmatrix} \boldsymbol{x}_0^T(t) & \dots & \boldsymbol{x}_{L-1}^T(t) \end{bmatrix}^T \\ &= \sum_{m=1}^{M} \mathbf{A}_m(t)\mathbf{d}_m + \begin{bmatrix} \boldsymbol{v}_0^T(t) & \dots & \boldsymbol{v}_{L-1}^T(t) \end{bmatrix}^T \end{aligned} \tag{1}$$

where the $(2L) \times L$ matrix $\mathbf{A}_m(t) = \mathbf{I}_L \otimes (\alpha_m(t)\mathbf{u}_m)$ in which the operator $\otimes$ stands for the Kronecker product. The steering vector is defined by $\mathbf{d}_m = [1\ e^{i\omega_m}\ \dots\ e^{i(L-1)\omega_m}]^T$. Since the problem addressed herein is to derive the SRL based on the CRB for the proposed model, we first start by deriving the CRB for (1) in the case of $M = 2$ sources.

III. Deterministic Cramér-Rao Bound Derivation

In the remaining of the correspondence, we will use the following assumptions:

A1. The noise is assumed to be a complex circular white Gaussian random noise with zero-mean and unknown variance σ^2. In addition, it is assumed to be both temporally and spatially uncorrelated.

A2. The sources are assumed to be known and deterministic (see, e.g., [20]–[24]). The unknown parameter vector is then given by[3] $\boldsymbol{\xi} = [\omega_1\ \omega_2\ \sigma^2]^T$.

A3. Furthermore, from a modelling point of view, we can assume, without loss of generality [5], that $L_{sd} = 2\pi A_{sl}/\lambda = 1$.

Using A1, the joint probability density function of the full observation vector $\boldsymbol{\chi} = [\boldsymbol{y}^T(1)\ \dots\ \boldsymbol{y}^T(N)]^T$ given $\boldsymbol{\xi}$ can be written as follows:

$$p(\boldsymbol{\chi}|\boldsymbol{\xi}) = \frac{1}{(\pi\sigma^2)^{2NL}} e^{\frac{-1}{\sigma^2}(\boldsymbol{\chi}-\boldsymbol{\mu})^H(\boldsymbol{\chi}-\boldsymbol{\mu})}$$

where $\boldsymbol{\mu} = \sum_{m=1}^{2} [\mathbf{d}_m^T \mathbf{A}_m(1)^T\ \dots\ \mathbf{d}_m^T \mathbf{A}_m(N)^T]^T$. Let $E\{(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})(\hat{\boldsymbol{\xi}} - \boldsymbol{\xi})^T\}$ be the covariance matrix of an unbiased estimator of $\boldsymbol{\xi}$, denoted by $\hat{\boldsymbol{\xi}}$. The covariance inequality principle states that under quite general/weak conditions $\text{MSE}([\hat{\boldsymbol{\xi}}]_i) = E\{([\hat{\boldsymbol{\xi}}]_i - [\boldsymbol{\xi}]_i)^2\} \geq \text{CRB}([\boldsymbol{\xi}]_i)$, where $\text{CRB}([\boldsymbol{\xi}]_i) = [\mathbf{FIM}^{-1}(\boldsymbol{\xi})]_{i,i}$ in which $\mathbf{FIM}(\boldsymbol{\xi})$ denotes the Fisher information matrix (FIM) regarding to the vector parameter $\boldsymbol{\xi}$.

Since we are working with a Gaussian observation model (assumption A1), the i^{th}, j^{th} element of the FIM for the parameter vector $\boldsymbol{\xi}$ can be written as [25]

$$[\mathbf{FIM}(\boldsymbol{\xi})]_{i,j} = \frac{NL}{\sigma^4}\frac{\partial\sigma^2}{\partial[\boldsymbol{\xi}]_i}\frac{\partial\sigma^2}{\partial[\boldsymbol{\xi}]_j} + \frac{2}{\sigma^2}\Re\left\{\frac{\partial\boldsymbol{\mu}^H}{\partial[\boldsymbol{\xi}]_i}\frac{\partial\boldsymbol{\mu}}{\partial[\boldsymbol{\xi}]_j}\right\}$$

where $(i,j) \in \{1,2,3\}^2$. $[\boldsymbol{\xi}]_i$ and $\Re\{z\}$ denote the i^{th} element of $\boldsymbol{\xi}$ and the real part of z, respectively. Then, the FIM for the proposed model is block-diagonal

$$\mathbf{FIM}(\boldsymbol{\xi}) = \frac{2}{\sigma^2}\begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0}^T & \frac{NL}{2\sigma^2} \end{bmatrix} \tag{2}$$

where

$$\begin{aligned} [\mathbf{F}]_{m,p} &= \Re\left\{\frac{\partial\boldsymbol{\mu}^H}{\partial\omega_m}\frac{\partial\boldsymbol{\mu}}{\partial\omega_p}\right\} \\ &= N\Re\left\{r_N\left(\mathbf{u}_m^H\mathbf{u}_p\mathbf{d}_m^H\mathbf{D}^2\mathbf{d}_p + K_{mp}\right)\right\}, \quad (m,p) \in \{1,2\}^2 \end{aligned} \tag{3}$$

in which $\mathbf{D} = \text{diag}\{0, \dots, L-1\}$, $r_N = (1/N)\sum_{t=1}^{N}\alpha_1^*(t)\alpha_2(t)$ and

$$K_{mp} = \frac{\partial\mathbf{u}_m^H}{\partial\omega_m}\frac{\partial\mathbf{u}_p}{\partial\omega_p}\mathbf{d}_m^H\mathbf{d}_p - i\mathbf{u}_m^H\frac{\partial\mathbf{u}_p}{\partial\omega_p}\mathbf{d}_m^H\mathbf{D}\mathbf{d}_p + i\frac{\partial\mathbf{u}_m}{\partial\omega_m}\mathbf{u}_p^H\mathbf{d}_m^H\mathbf{D}\mathbf{d}_p.$$

Using the fact that the polarization state vector of a COLD array is not a function of the direction parameter, thus $\partial\mathbf{u}_m/\partial\omega_m = \mathbf{0}$. Consequently $K_{mp} = 0$ and (3) becomes $[\mathbf{F}]_{mp} = N\Re\{r_N\mathbf{u}_m^H\mathbf{u}_p\mathbf{d}_m^H\mathbf{D}^2\mathbf{d}_p\}$. Furthermore, as $\|\mathbf{u}_m\|^2 = 1$, one obtains $[\mathbf{F}]_{i,i} = Na_i^2\alpha$ for $i = 1, 2$ where $\alpha = (1/6)(L-1)L(2L-1)$. The cross terms are given by $[\mathbf{F}]_{1,2} = [\mathbf{F}]_{2,1} = N\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\}$ where $\mathbf{u}_1^H\mathbf{u}_2 = \cos(\rho_1)\cos(\rho_2) + \sin(\rho_1)\sin(\rho_2)e^{i(\psi_2-\psi_1)}$ and

$$\eta = \sum_{\ell=0}^{L-1}\ell^2 e^{-i(\omega_1-\omega_2)\ell} = \sum_{\ell=0}^{L-1}\ell^2 e^{-i\,\text{sgn}(\omega_1-\omega_2)\delta_\omega^{(\text{COLD})}\ell} \tag{4}$$

in which $\delta_\omega^{(\text{COLD})} = |\omega_1 - \omega_2|$ and $\text{sgn}(z) = z/|z|$ for $z \neq 0$. To simplify the derivations and without loss of generality, we choose $\omega_1 > \omega_2$ in the following. Consequently, the inverse of the FIM is given by

$$\mathbf{F}^{-1} = \frac{N}{\det\{\mathbf{F}\}}\begin{bmatrix} a_2^2\alpha & -\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\} \\ -\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\} & a_1^2\alpha \end{bmatrix} \tag{5}$$

where $\det\{\mathbf{F}\} = N^2(a_1^2a_2^2\alpha^2 - \Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\})$. Finally, replacing (2) and (5) into $\mathbf{CRB}(\boldsymbol{\xi}) = \mathbf{FIM}^{-1}(\boldsymbol{\xi})$, the CRBs (see Fig. 1) are given by

$$\text{CRB}(\omega_1) \triangleq [\mathbf{F}^{-1}]_{1,1} = \frac{\sigma^2}{2N}\frac{a_2^2\alpha}{a_1^2a_2^2\alpha^2 - \Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\}} \tag{6}$$

$$\text{CRB}(\omega_2) \triangleq [\mathbf{F}^{-1}]_{2,2} = \frac{\sigma^2}{2N}\frac{a_1^2\alpha}{a_1^2a_2^2\alpha^2 - \Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\}} \tag{7}$$

$$\text{CRB}(\omega_1,\omega_2) \triangleq [\mathbf{F}^{-1}]_{1,2} = -\frac{\sigma^2}{2N}\frac{\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\}}{a_1^2a_2^2\alpha^2 - \Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\eta\}}. \tag{8}$$

IV. Statistical Resolution Limit

This section is devoted to the derivation of the SRL of the COLD-ULA. Taking advantage of the previously derived CRBs (6), (7), and

[1]One should note that due to the nature of the COLD array sensors, one has twice the number of measurements w.r.t. a ULA array with the same number of sensors and the same array's aperture.

[2]Note that this source model is commonly used in many digital communication systems (see [5] and references therein).

[3]Note that the state parameter vector is assumed to be known. However, this assumption is not severe (since the numerical simulations part).

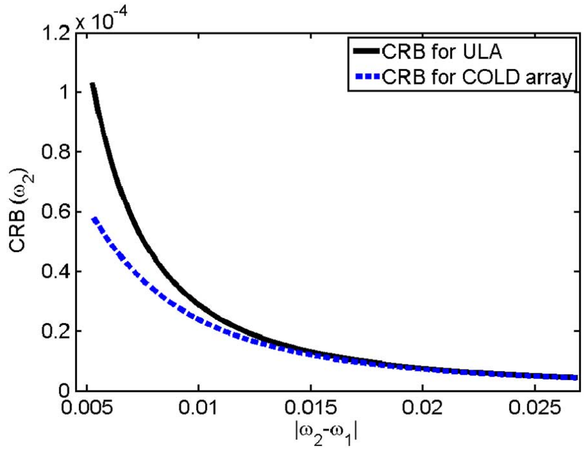


Fig. 1. The CRB for the COLD array and the ULA with $N = 100$ snapshots and $L = 25$ sensors. One can notice that for a small separation the CRB for the ULA goes to infinity faster than the CRB for the COLD array. This can be explained by the additional knowledge about polarization parameters in the case of the COLD array.

(8), the SRL in the Smith sense is derived in Section IV-A. Then, the SRL based on a hypothesis test is deduced in Section IV-B. One should note that the SRL of the ULA according to the model (1) is not derived in the literature. This latter can be derived following the same steps as in Section IV-A leading to

$$\delta_\omega^{(\mathrm{ULA})} = \frac{\sigma}{\sqrt{2N\alpha}}\sqrt{\frac{a_1^2 + a_2^2 + 2\Re\{r_N\}}{a_1^2 a_2^2 - \Re^2\{r_N\}}}.$$

A. Statistical Resolution Limit for a COLD-ULA

Let $\delta_\omega^{(\mathrm{COLD})}$ denoting the SRL of COLD-ULA according to the model (1). Thus, one obtains (see [19])

$$\mathrm{CRB}\left(\delta_\omega^{(\mathrm{COLD})}\right) = \mathrm{CRB}(\omega_1) + \mathrm{CRB}(\omega_2) - 2\mathrm{CRB}(\omega_1, \omega_2). \quad (9)$$

Consequently, the SRL[4] is defined as the minimal separation, denoted $\delta_\omega^{(\mathrm{COLD})}$, which resolves the following implicit equation:

$$\delta_\omega^{(\mathrm{COLD})} = \sqrt{\mathrm{CRB}\left(\delta_\omega^{(\mathrm{COLD})}\right)} \Longleftrightarrow f\left(\delta_\omega^{(\mathrm{COLD})}\right) = A \quad (10)$$

where $f(\delta_\omega^{(\mathrm{COLD})}) = (2/\sigma^2)\det\{\mathbf{F}\}\left((\delta_\omega^{(\mathrm{COLD})})^2 + 2\mathrm{CRB}(\omega_1, \omega_2)\right)$ and $A = (a_1^2 + a_2^2)\alpha$. In the following, (10) is solved to obtain the desired SRL for the orthogonal and non-orthogonal signal sources cases.

1) The Orthogonal Signal Sources Case: In the case of orthogonal signal sources [20], one has $r_N = (1/N)\sum_{t=1}^{N} \alpha_1^*(t)\alpha_2(t) = 0$. This implies that the FIM is diagonal (i.e., the parameters of interest are decoupled). Thus, replacing $\mathrm{CRB}(\omega_1, \omega_2) = 0$ and $r_N = 0$ into (10), the SRL in the orthogonal signal sources case, denoted by $\delta_\omega^{(\mathrm{COLD-O})}$, is given by

$$\delta_\omega^{(\mathrm{COLD-O})} = \frac{\sigma}{\sqrt{2N\alpha}}\sqrt{\frac{a_1^2 + a_2^2}{a_1^2 a_2^2}}. \quad (11)$$

It can be readily checked that the SRL is not a function of the polarization parameters. Consequently, in comparison to the classical ULA array, the use of the COLD array cannot improve the resolvability of the sources in this scenario. Moreover, for equipowered sources $(a_1 = a_2 = a)$, one obtains

$$\delta_\omega^{(\mathrm{COLD-O})} = \frac{1}{\sqrt{N\alpha\mathrm{SNR}}} \quad (12)$$

where $\mathrm{SNR} = a^2/\sigma^2$. Furthermore, for equipowered sources and a large number of sensors $(L \gg 1)$, the SRL can be approximated by $\delta_\omega^{(\mathrm{COLD-O})} \approx \sqrt{3}/(N^{1/2}\mathrm{SNR}^{1/2}L^{3/2})$. Note that, in this case, the SRL is proportional to the inverse square root of the number of snapshots, to the inverse square root of the SNR and to inverse of $L\sqrt{L}$. Also note that, the SRL obtained here is qualitatively consistent with the SRL derived in [12], [17] in the case of a classical ULA array.

2) The Non-Orthogonal Signal Sources Case: The analysis in the general case of non-orthogonal signal sources (i.e., $r_N \neq 0$) is more complex and needs some approximations. Considering the second-order Taylor expansion of the functional η (see (4)) around $\delta_\omega^{(\mathrm{COLD})} = 0$, one obtains, for $L\delta_\omega^{(\mathrm{COLD})} \ll 1$, $\eta \approx \sum_{\ell=0}^{L-1} \ell^2(1 + i\delta_\omega^{(\mathrm{COLD})}\ell) = \alpha + i\beta\delta_\omega^{(\mathrm{COLD})}$, where $\beta = \sum_{\ell=0}^{L-1} \ell^3 = (1/4)(L-1)^2L^2$ (note that this approximation is not severe, since numerical simulation shows that the SRL based on the second-order Taylor expansion of η is close, and in a good agreement with the exact SRL; see Fig. 3). One can note that expression (10), for non-orthogonal signal sources, becomes, for $L\delta_\omega^{(\mathrm{COLD})} \ll 1$,

$$\left(\delta_\omega^{(\mathrm{COLD})}\right)^2 = \frac{\sigma^2}{2N}\frac{A + 2B - 2\delta_\omega^{(\mathrm{COLD})}\bar{B}}{C^2 - \left(B - \delta_\omega^{(\mathrm{COLD})}\bar{B}\right)^2} \quad (13)$$

where $B = \alpha\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\}$, $\bar{B} = \beta\Im\{r_N\mathbf{u}_1^H\mathbf{u}_2\}$ and $C = a_1a_2\alpha$ in which $\Im\{z\}$ denote the imaginary part of z. Expression (13) is in fact the roots of the following polynomial

$$p(x) = 2N\bar{B}x^4 + 4NB\bar{B}x^3 + 2N(B^2 - C^2)x^2 - 2\sigma^2\bar{B}x + \sigma^2(A + 2B) \quad (14)$$

where $x = \delta_\omega^{(\mathrm{COLD})}$.

Resolving this polynomial can be facilitated by noticing that, if $\delta_\omega^{(\mathrm{COLD})}$ is a root then $-\delta_\omega^{(\mathrm{COLD})}$ is also a root.[5] Consequently, $p(x)$ can be rewritten as $p(x) = (x - \delta_\omega^{(\mathrm{COLD})})(x + \delta_\omega^{(\mathrm{COLD})})(x - s_1)(x - s_2)$, where s_1 and s_2 are the unwanted roots. From the latter expression and (14), one obtains

$$\begin{cases} s_1 + s_2 = 2NB \\ s_1s_2 - \left(\delta_\omega^{(\mathrm{COLD})}\right)^2 = \frac{(B^2 - C^2)}{B} \\ \left(\delta_\omega^{(\mathrm{COLD})}\right)^2(s_1 + s_2) = \frac{-\sigma^2}{N} \\ -\left(\delta_\omega^{(\mathrm{COLD})}\right)^2 s_1s_2 = \frac{\sigma^2(A+2B)}{2N\bar{B}}. \end{cases} \quad (15)$$

Using the second and last equation of (15) one obtains the SRL as the root (we keep only the positive root whatever the sign of $\bar{B}$) of $2N\bar{B}(\delta_\omega^{(\mathrm{COLD})})^4 + 2N(B^2 - C^2)(\delta_\omega^{(\mathrm{COLD})})^2 + \sigma^2(A + 2B) = 0$. Consequently,

$$(\delta_\omega^{(\mathrm{COLD})})^2 = \frac{C^2 - B^2}{2\bar{B}}\left(1 - \sqrt{1 - 2\sigma^2\frac{(A + 2B)}{(B^2 - C^2)^2}\frac{\bar{B}}{N}}\right).$$

[4]From (9), one should note that the SRL using the Smith criterion [16] takes into account the coupling between the parameters of interest unlike the Lee criterion [13], see Fig. 2 (*right*).

[5]Indeed, using the change variable formula (see [26] p. 45) w.r.t. $\tilde{\delta} = -\delta_\omega^{(\mathrm{COLD})}$, the Jacobian matrix $\mathbf{J}$ is reduced to a scalar $J = -1$. Thus, $\mathrm{CRB}(-\delta_\omega^{(\mathrm{COLD})}) = \mathrm{CRB}(\tilde{\delta}) = J^2\mathrm{CRB}(\delta_\omega^{(\mathrm{COLD})}) = \mathrm{CRB}(\delta_\omega^{(\mathrm{COLD})})$. Consequently, if $\delta_\omega^{(\mathrm{COLD})}$ is a root of $(\delta_\omega^{(\mathrm{COLD})})^2 = \mathrm{CRB}(\delta_\omega^{(\mathrm{COLD})})$ then $-\delta_\omega^{(\mathrm{COLD})}$ is also a root.

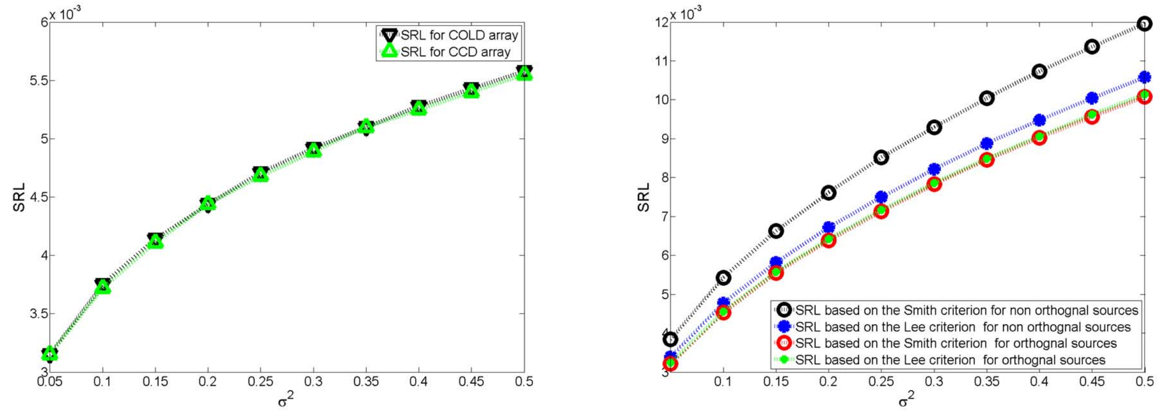


Fig. 2. Left: The SRL using a COLD and a cocentered crossed-dipole (CCD) [5]. One notices that the SRL of a CCD array is in a good agreement with the SRL of a COLD array. However, the SRL closed form expression of the COLD array is easier to derive since the COLD array exhibits some interesting properties, as for instance, the insensibility of the polarization vector to the source localization in the plan of the array and the constant norm of the polarization vector. Right: The SRL based on the Smith and Lee criterion. One can notice that in the case of orthogonal signal sources, the SRL based on the Smith and Lee criterion coincides (upon a normalization factor). However, in the general case (i.e., not orthogonal signal sources) the Lee criterion, unlike the Smith criterion, ignores the coupling terms between the parameters of interest.

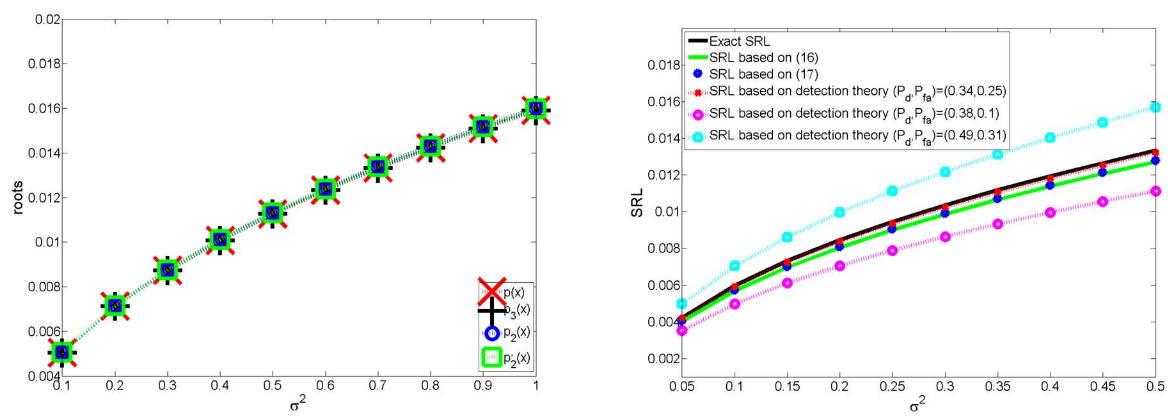


Fig. 3. Left: Illustration of the desired roots of the polynomials $p_2(x), p_2'(x), p_3(x)$ and $p_4(x)$. Right: Comparison with literature results: The SRL versus σ^2 for $N = 100$: the approximated SRL based on (16) and (17) is in good agreement with the exact SRL (i.e., the numerical solution of (10) without any approximation). This validate the closed-form expressions given in (16) and (17). Furthermore, one can notice that, for example, for $P_d = 0.37$ and $P_{\rm fa} = 0.1$ the SRL based on the SRL (16) and (17) is almost equal to the SRL based on a hypothesis test [11] derived in the asymptotic case. From the case $P_d = 0.49$ and $P_{\rm fa} = 0.3$ or/and $P_d = 0.32$ and $P_{\rm fa} = 0.1$, one can notice the influence of the translation factor ρ on the SRL.

One should note that under realistic conditions $\delta_\omega^{(\mathrm{COLD})}$ exists since $\frac{(A+2B)}{(B^2-C^2)^2}\frac{\bar{B}}{N}$ is $o(\frac{1}{NL^5})$ (i.e., $|\frac{(A+2B)}{(B^2-C^2)^2}\frac{\bar{B}}{N}| \ll 1$). Consequently, the desired SRL is given by (we discard the negative root) (see (16), shown at the bottom of the page). Note that, unlike the orthogonal signal sources case, the SRL depends on the state vector parameter.

Remark 1: Note that the latter formula is valid if $\bar{B} \neq 0$. When $\bar{B} = 0$, the roots of $p(x)$ (which become the roots of $p_2(x) \triangleq 2N(B^2 - C^2)x^2 + \sigma^2(A + 2B))$ are given by $x^2 = \frac{-\sigma^2(A+2B)}{2N(B^2-C^2)}$. The real root exists if in particular $C^2 - B^2 > 0$ and $A + 2B \geq 0$. Since $|\Re\{xy\}| \leq |xy| \leq |x||y|$, where $|.|$ denotes the absolute value of a real number or the modulus of a complex number, then, for a fixed value of t, one has $|\Re\{e^{i(\phi_2(t)-\phi_1(t))}\mathbf{u}_1^H\mathbf{u}_2\}| \leq |e^{j(\phi_2(t)-\phi_1(t))}||\mathbf{u}_1^H\mathbf{u}_2| \leq 1$. Thus, $|\Re\{\sum_{t=1}^N e^{i(\phi_2(t)-\phi_1(t))}\mathbf{u}_1^H\mathbf{u}_2\}| \leq \sum_{t=1}^N |\Re\{e^{i(\phi_2(t)-\phi_1(t))}\mathbf{u}_1^H\mathbf{u}_2\}| \leq N \leq N\frac{a_1^2+a_2^2}{2a_1a_2}$. Consequently, $A \geq -2B$ is satisfied. On the other hand, since $\Im\{r_N\mathbf{u}_1^H\mathbf{u}_2\} = 0$, thus, $|\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\}| = |r_N||\mathbf{u}_1^H\mathbf{u}_2|$. Assuming different polarization state vectors, i.e., $(\rho_1,\psi_1) \neq (\rho_2,\psi_2)$, one obtains $|r_N||\mathbf{u}_1^H\mathbf{u}_2| < |r_N| \leq (a_1a_2/N)\sum_{t=1}^N |e^{i(\phi_2(t)-\phi_1(t))}| = a_1a_2$. Thus, $B^2 < C^2$. Finally, one concludes that the root of (14) in

$$\begin{aligned}\delta_\omega^{(\mathrm{COLD})} &= \sqrt{\frac{C^2-B^2}{2\bar{B}}\left(1-\sqrt{1-2\sigma^2\frac{(A+2B)}{(B^2-C^2)^2}\frac{\bar{B}}{N}}\right)}\\ &= \alpha\sqrt{\frac{a_1^2a_2^2-\Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\}}{2\beta\Im\{r_N\mathbf{u}_1^H\mathbf{u}_2\}}\left(1-\sqrt{1-\frac{2\sigma^2\beta\Im\{r_N\mathbf{u}_1^H\mathbf{u}_2\}}{\alpha N}\frac{((a_1^2+a_2^2)+2\Re\{r_N\mathbf{u}_1^H\mathbf{u}_2\})}{(\Re^2\{r_N\mathbf{u}_1^H\mathbf{u}_2\}-a_1^2a_2^2)^2}}\right)}.\end{aligned} \tag{16}$$

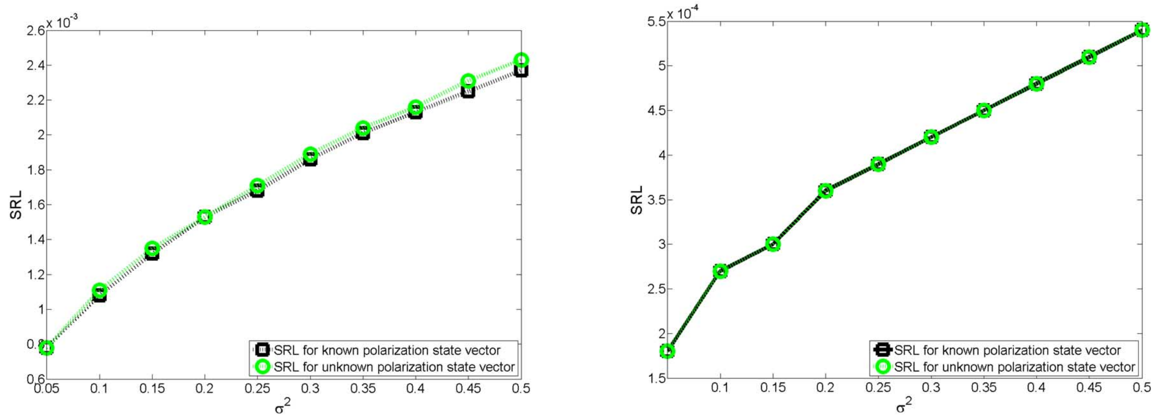


Fig. 4. The SRL of the COLD-ULA with (left) $N = 40$ snapshots and $L = 5$ sensors, and (right) $N = 100$ snapshots and $L = 10$ sensors. Note that the SRL using the assumption of known state parameter vector is almost identical to the SRL of unknown state parameter vector.

the case $\Im\{r_N \mathbf{u}_1^H \mathbf{u}_2\} = 0$ exists and is given by (we discard the negative root)

$$\delta_\omega^{(\mathrm{COLD})} = \frac{\sigma}{\sqrt{2N\alpha}} \sqrt{\frac{a_1^2 + a_2^2 + 2\Re\{r_N \mathbf{u}_1^H \mathbf{u}_2\}}{a_1^2 a_2^2 - \Re^2\{r_N \mathbf{u}_1^H \mathbf{u}_2\}}}. \tag{17}$$

Remark 2: From Fig. 3 (left), one can notice that the desired roots of $p_4(x), p_3(x) \triangleq 4NB\bar{B}x^3 + 2N(B^2 - C^2)x^2 - 2\sigma^2\bar{B}x + \sigma^2(A+2B)$, $p_2(x)$ and $p_2'(x) \triangleq 2N(B^2 - C^2)x^2 - 2\sigma^2\bar{B}x + \sigma^2(A+2B)$ are almost identical for various values of σ^2. Indeed, this is expected since the desired roots, corresponding to the SRL, are small (i.e., $\delta_\omega^{(\mathrm{COLD})} \ll 1$). Furthermore, for a sufficient number of sensors, the coefficient corresponding to the fourth, third and first degree of the polynomial $p(x)$ are small (i.e., $2N\bar{B} \sim O(1)$, $4NB\bar{B} \sim O(1/N)$ and $2\sigma^2\bar{B} \sim O(1/N)$ whereas $2N(B^2 - C^2) \sim O(N)$).

Remark 3: On the other hand, since $|\frac{(A+2B)}{(B^2-C^2)^2}\frac{\bar{B}}{N}| \ll 1$, the second-order Taylor expansion of (16) around $\frac{(A+2B)}{(B^2-C^2)^2}\frac{\bar{B}}{N} = 0$ gives

$$\delta_\omega^{(\mathrm{COLD})} = \frac{\sigma}{\sqrt{2N\alpha}} \sqrt{\frac{a_1^2 + a_2^2 + 2\Re\{r_N \mathbf{u}_1^H \mathbf{u}_2\}}{a_1^2 a_2^2 - \Re^2\{r_N \mathbf{u}_1^H \mathbf{u}_2\}}} \tag{18}$$

which is the same expression as in (17). Furthermore, for orthogonal signal sources, one obtains (11). Consequently, (17) unifies the different cases of the SRL derivation results.

Remark 4: Finally, using (17) and for equipowered sources (i.e., $a_1 = a_2 = a$), one obtains

$$\delta_\omega^{(\mathrm{COLD})} = \frac{1}{\sqrt{N\alpha\mathrm{SNR}}} \sqrt{\frac{1 + \Re\{\tilde{r}_N \mathbf{u}_1^H \mathbf{u}_2\}}{1 - \Re^2\{\tilde{r}_N \mathbf{u}_1^H \mathbf{u}_2\}}} \tag{19}$$

in which $\tilde{r}_N = (1/N)\sum_{t=1}^N e^{j(\phi_2(t) - \phi_1(t))}$. Note that, the SRL obtained in (19) is qualitatively consistent with the SRL derived in [12] and [17] in the case of a classical ULA array.

Remark 5: The polarization state vector of a COLD array is not function of the direction parameter (i.e., $\partial \mathbf{u}_m / \partial \omega_m = \mathbf{0}$ for $m = 1, 2$). Remark that this is not the case for Cocentered Crossed-Dipole (CCD) antenna. This nice property of the COLD array allows to greatly simplify the analysis of the SRL, see Fig. 2 (left) for a comparison between the CCD-ULA and the COLD-ULA SRL.

Furthermore, from A2, one can note that the state parameter vector is assumed to be known. However, this assumption is not severe, since numerical simulations show that the SRL of a known state parameter vector is close to the SRL of a unknown state parameter vector (even for a low number of sensors $L = 5$ and/or a low number of snapshots $N = 40$); see Fig. 4.

B. SRL Based on a Hypothesis Test

Another approach to derive the SRL is based on a hypothesis test. In this Subsection, we show that the results of [11] in the case of non-polarized sources can be extended to the polarized sources case. Indeed, using a binary hypothesis test and the same method as in [11], the asymptotic (in terms of snapshots) SRL based on a hypothesis test is given as the root of (proof: see the Appendix)

$$\delta_{\mathrm{detection}} = \rho\sqrt{\mathrm{CRB}(\delta_{\mathrm{detection}})}. \tag{20}$$

The so-called translation factor, ρ, is determined numerically, for a given probability of detection P_d and a given probability of false alarm P_{fa}, as the root of $\mathcal{Q}^{-1}_{\chi'^2_2(\rho)}(P_d) = \mathcal{Q}^{-1}_{\chi^2_2}(P_{\mathrm{fa}})$. In which $\mathcal{Q}^{-1}_{\chi^2_2}(.)$ and $\mathcal{Q}^{-1}_{\chi'^2_2(\rho)}(.)$ denote the inverse of the right tail probability of the central chi-squared pdf χ^2_2 and the noncentral chi-squared pdf $\chi'^2_2(\rho)$, respectively).

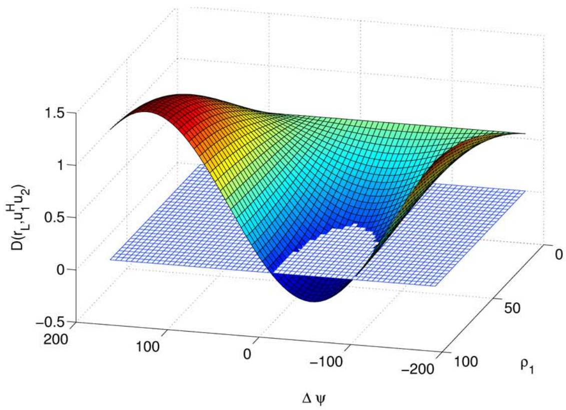
Remark 6: The hypothesis test used to derive (20) is a binary one-sided test and the MLE used is an unconstrained estimator (see the Appendix), thus, one can deduce that the GLRT, used to derive the asymptotic SRL, is [27] 1) asymptotically uniformly most powerful (UMP) test among all invariant statistical tests, and 2) has asymptotic constant false-alarm rate (CFAR).

Fig. 3 (right) shows that the derived SRL (17) is in agreement, with respect to the translation factor, with the extension of the SRL based on a UMP and CFAR hypothesis test in the asymptotic case, which assesses the validity of our results. In addition, this figure shows that the derived SRL is tight w.r.t. the exact SRL (i.e., the numerical solution of (10) without any approximation). Furthermore, Fig. 3 (right) assesses remark 2 and 3 since the SRL (17) derived using $p_2(x)$ is almost identical to the SRL (16) derived using $p(x)$.

In the following section, a comparison between the SRL of two polarized sources impinging on a COLD-ULA and on an ULA, is done.

V. Comparison Between the Statistical Resolution Limit of a COLD-ULA and an ULA and Numerical Analysis

Consider two radiating far-field and narrowband sources observed by a classical ULA of L sensors with interelement spacing d [25]. The array and the emitted signals are coplanar. Following the same steps leading to $\delta_\omega^{(\mathrm{COLD-O})}$, one obtains after some algebra calculations the SRL of the ULA denoted by $\delta_\omega^{(\mathrm{ULA-O})}$. The derivations are not reported here since they are similar to the ones presented for the COLD array. As in Section IV, we detail the orthogonal and non-orthogonal signal sources case.



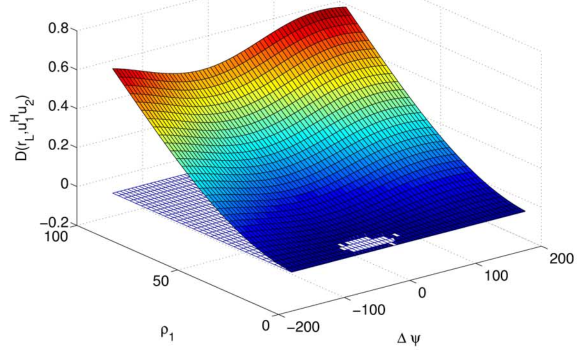


Fig. 5. $D(r_L, \mathbf{u}_1^H \mathbf{u}_2)$ versus the polarization state parameters ρ and ψ; $a_1 = 2, a_2 = 3, r_N = (1+i)/20$ where $N = 20$. (Left) $\rho_2 = 85°$ and (right) $\rho_2 = 5°$.

A. Comparison in the Orthogonal Signal Sources Case

In the case where the signal sources are orthogonal (i.e., $r_N = 0$ [20]), one obtains (after calculus) $\delta_\omega^{(\mathrm{ULA-O})} = \delta_\omega^{(\mathrm{COLD-O})}$ meaning that the COLD-ULA and the classical ULA have the same resolvability capacity.

B. Comparison in the Non-Orthogonal Signal Sources Case

In the following we focus on the SRL given by (18) (see remark 3). After calculus, one obtains the SRL of the ULA

$$\delta_\omega^{(\mathrm{ULA})} = \frac{\sigma}{\sqrt{2N\alpha}}\sqrt{\frac{(a_1^2 + a_2^2) + 2\Re\{r_N\}}{a_1^2 a_2^2 - \Re^2\{r_N\}}}. \tag{21}$$

Thus, from (17) and (21), one can check that

$$\delta_\omega^{(\mathrm{COLD})} \leq \delta_\omega^{(\mathrm{ULA})} \quad \text{iff} \quad \Re\{r_N\} \geq \Re\left\{r_N \mathbf{u}_1^H \mathbf{u}_2\right\}. \tag{22}$$

As $\Re\{r_N \mathbf{u}_1^H \mathbf{u}_2\} = \Re\{r_N\}\Re\{\mathbf{u}_1^H \mathbf{u}_2\} - \Im\{r_N\}\Im\{\mathbf{u}_1^H \mathbf{u}_2\}$ and $\Re\{\mathbf{u}_1^H \mathbf{u}_2\} \leq 1$, condition (22) is satisfied for $\Im\{r_N\} = 0$ or/and $\Im\{\mathbf{u}_1^H \mathbf{u}_2\} = 0$. Consequently, we have $\delta_\omega^{(\mathrm{COLD})} < \delta_\omega^{(\mathrm{ULA})}$ for the following cases:

C1. if the signals are real and positive, i.e., $\Im\{r_N\} = 0$ or with the same phase, i.e., $\phi_1(t) = \phi_2(t), \forall t$;

C2. if $\psi_1 = \psi_2$, i.e., $\Im\{\mathbf{u}_1^H \mathbf{u}_2\} = 0$;

C3. if $\rho_1 = 0$ or $\rho_2 = 0$, i.e., $\Im\{\mathbf{u}_1^H \mathbf{u}_2\} = 0$.

Besides C1., C2., and C3., in Fig. 5 we plot $D(r_L, \mathbf{u}_1^H \mathbf{u}_2) = \Re\{r_N\} - \Re\{r_N \mathbf{u}_1^H \mathbf{u}_2\}$ versus the polarization state parameters ρ and ψ. Consequently, from (22) if $D > 0$ thus $\delta_\omega^{(\mathrm{COLD})} < \delta_\omega^{(\mathrm{ULA})}$. Fig. 5 suggests that generally $\delta_\omega^{(\mathrm{COLD})} < \delta_\omega^{(\mathrm{ULA})}$ while $\delta_\omega^{(\mathrm{COLD})} > \delta_\omega^{(\mathrm{ULA})}$ only for a small region (which corresponds to the part of the plot that is under the horizontal plan). This means that generally, the SRL of the COLD-ULA is smaller than the one for the ULA.

VI. Conclusion

In this correspondence, we derived the deterministic CRB in a non-matrix closed form expression for two polarized far-field time-varying narrowband known sources observed by a COLD-ULA. Taking advantage of these expressions, we deduced the SRL for the COLD-ULA which was compared to the SRL of the ULA. We noticed that, surprisingly, in the case where the signal sources are orthogonal, the SRL of the COLD-ULA is equal to the SRL of the ULA, meaning that it is not a function of polarization parameters. Furthermore, for non-orthogonal signal sources, we have given three sufficient and necessary conditions such that the SRL of the COLD-ULA is less than the SRL of the ULA. By analytical expressions and numerical simulations we have shown that the SRL of the ULA is less than the SRL of the COLD-ULA only for few cases, meaning that generally the performance of the COLD-ULA is better than the performance of the ULA. Note that an interesting work could be to apply the proposed method in the case of Gaussian sources and to compare it to [17, eq.(9)].

Appendix

Let us consider the following binary hypothesis test where $\mathcal{H}_0$ and $\mathcal{H}_1$ represent the presence of one signal and the presence of two signals, respectively. Consequently, following the same line as in [11], one can formulate the hypothesis test, as a simple one-sided binary hypothesis test as follows:

$$\begin{cases} \mathcal{H}_0: & \delta_{\mathrm{detection}} = 0 \\ \mathcal{H}_1: & \delta_{\mathrm{detection}} > 0 \end{cases} \tag{23}$$

where $\delta_{\mathrm{detection}}$ denotes the SRL based on a hypothesis test such that $\delta_{\mathrm{detection}} = |\omega_1 - \omega_2|$. To derive the SRL based on a hypothesis test, we consider the GLRT [27]:

$$L_G(\boldsymbol{y}) = \frac{p(\boldsymbol{y}|\hat{\delta}_{\mathrm{detection}}, \hat{\sigma}_1, \mathcal{H}_1)}{p(\boldsymbol{y}|\hat{\sigma}_0, \mathcal{H}_0)} >^{\mathcal{H}_1} \varsigma' \tag{24}$$

where $\hat{\delta}_{\mathrm{detection}}$, $\hat{\sigma}_1$ and $\hat{\sigma}_0$ denote the maximum likelihood estimates (MLE) of $\delta_{\mathrm{detection}}$ under $\mathcal{H}_1$, the MLE of σ under $\mathcal{H}_1$ and the MLE of σ_0 under $\mathcal{H}_0$, respectively, in which ς' denotes the test threshold (the central spatial phase factor is implicitly assumed unknown). From (24), one obtains

$$T_G(\boldsymbol{y}) = \mathrm{Ln} L_G(\boldsymbol{y}) >^{\mathcal{H}_1} \varsigma = \mathrm{Ln}\varsigma'. \tag{25}$$

Deriving and analyzing the SRL from (25) seems to be hard and even intractable in some cases (especially due to the derivation of $\hat{\delta}_{\mathrm{detection}}$). Consequently, in the following we consider the asymptotic case. In [27] it has been proved that, for a large number of snapshots, the statistic $T_G(\boldsymbol{y})$ in (25) follows:

$$T_G(\boldsymbol{y}) \sim \begin{cases} \chi_2^2 & \text{under } \mathcal{H}_0 \\ {\chi'}_2^2(\rho') & \text{under } \mathcal{H}_1 \end{cases} \tag{26}$$

where χ_2^2 and ${\chi'}_2^2(\rho')$ denote the central chi-square pdf and the noncentral chi-square pdf both with two degrees of freedom. The noncentral parameter ρ' is given by [27]

$$\rho' = \hat{\delta}_{\mathrm{detection}}^2 \left(\mathrm{CRB}(\delta_{\mathrm{detection}})\right)^{-1}. \tag{27}$$

Since we consider the asymptotic case $\hat{\delta}_{\mathrm{detection}} \approx \delta_{\mathrm{detection}}$, thus (27) becomes $\delta_{\mathrm{detection}}^2 = \rho' \mathrm{CRB}(\delta_{\mathrm{detection}})$. Consequently, $\delta_{\mathrm{detection}} =$

$\rho\sqrt{\mathrm{CRB}(\delta_{\mathrm{detection}})}$ where $\sqrt{\rho} = \rho'$ represents the so-called translation factor [11] which is determined due to the probability of detection P_d and the probability of false alarm P_{fa} as follows: $P_{\mathrm{fa}} = \mathcal{Q}_{\chi_2^2}(\varsigma)$ and $P_d = \mathcal{Q}_{\chi'^2_2(\rho^2)}(\varsigma)$ where $\mathcal{Q}_{\chi_2^2}(.)$ and $\mathcal{Q}_{\chi'^2_2(\rho^2)}(.)$ denote the right tail probability of χ_2^2 and $\chi'^2_2(\rho^2)$, respectively. This conclude the proof.

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A Barankin-Type Bound on Direction Estimation Using Acoustic Sensor Arrays

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***Abstract*—We derive a Barankin-type bound (BTB) on the mean-square error (MSE) in estimating the directions of arrival (DOAs) of far-field sources using acoustic sensor arrays. We consider narrowband and wideband deterministic source signals, and scalar or vector sensors. Our results provide an approximation to the threshold of the signal-to-noise ratio (SNR) below which the performance of the maximum likelihood estimation (MLE) degrades rapidly. For narrowband DOA estimation using uniform linear vector-sensor arrays, we show that this threshold increases with the distance between the sensors. As a result, for medium SNR values the performance does not necessarily improve with this distance.**

***Index Terms*—Acoustic sensor array, acoustic vector sensor, Barankin bound, direction of arrival estimation, threshold SNR.**

I. INTRODUCTION

The Barankin bound [1]–[4] is a useful tool in estimation problems for predicting the threshold region of signal-to-noise ratio (SNR) [5]–[8], below which the accuracy of the maximum likelihood estimation (MLE) degrades rapidly. Identification of the threshold region enables to determine the operation conditions, such as observation time and transmission power, to obtain a desired performance.

In the recent years, many works have been carried out for identification of the threshold region of the MLE. One approach is based on the method of interval estimation (MIE) [9] in which the performance of the MLE in the threshold region is approximated. However,

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