Statistical Resolution Limit for Source Localization With Clutter Interference in a MIMO Radar Context

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Abstract—During the last decade, multiple-input multiple-ouput (MIMO) radar has received an increasing interest. One can find several estimation schemes in the literature related to the direction of arrivals and/or direction of departures, but their ultimate performance in terms of the statistical resolution limit (SRL) have not been fully investigated. In this correspondence, we fill this lack. Particulary, we derive the SRL to resolve two closely spaced targets in clutter interference using a MIMO radar with widely separated antennas. Toward this end, we use a hypothesis test formulation based on the generalized likelihood ratio test (GLRT). Furthermore, we investigate the link between the SRL and the minimum signal-to-noise ratio (SNR) required to resolve two closely spaced targets for a given probability of false alarm and for a given probability of detection. Finally, theoretical and numerical analysis of the SRL are given for several scenarios (with/without clutter interference, known/unknown parameters of interest and known/unknown noise variance).

Index Terms—Clutter interference, MIMO radar, performance analysis, statistical resolution limit.

I. INTRODUCTION

Based on the attractive multiple-input multiple-ouput (MIMO) communication theory, the MIMO radar has received an increasing interest [1]. The advantage of the MIMO radar is to use multiple antennas to simultaneously transmit several noncoherent known waveforms and to exploit multiple antennas to receive the reflected signals (echoes).

One can find a plethora of algorithms for target localization using a MIMO radar and some related lower bounds (see [1]–[4] and references therein). However, their ultimate performance in terms of the statistical resolution limit (SRL) has not been fully investigated. The SRL [5]–[8], defined as the minimal separation between two signals in terms of the parameter of interest allowing a correct source resolvability, is an essential tool to quantify the estimator performance. Among all the different approaches to characterize the SRL, one can find three families. i) The first one is based on the null spectrum [9], [10]. However, this criterion is only relevant to a specific high-resolution algorithm. ii) The second one is based on the estimation accuracy [5], [11], [12]. Indeed, since the Cramér–Rao bound (CRB) expresses a lower bound on the covariance matrix of any unbiased estimator, then it expresses also the ultimate estimation accuracy. Consequently, it could be used to describe/obtain the SRL. For example, in this context, the Smith criterion states that two signals are resolvable if the separation (between the parameters of interest) is less than the standard deviation of the separation estimation [5]. iii) The last one is based on detection theory using a hypothesis test formulation [7], [8], [13]. The main idea is to decide if one or two closely spaced signals are present in the set of the observations. Consequently, in this context, the challenge is to link the minimum separation, between two targets, that is detectable at a given SNR (for a given probability of false alarm and a given probability of detection).

Several works have been done on the SRL and most of them in the context of spectral analysis and/or far field source localization ([5], [7]–[14] and the references therein). However, in the MIMO radar context, to the best of our knowledge, no results are available (except in [3] where one can find the asymptotic SRL, using the Smith criterion, for the co-located MIMO radar without clutter interference and with a prior knowledge on the target and the radar cross-section). The goal of this paper is to derive the SRL for two targets imbedded in clutter interference. We consider a MIMO radar with widely separated arrays (i.e., where the transmitter and the receiver are far enough so that they do not share the same angle variable [2], [4]). The cases of known/unknown parameters of interest and known/unknown nuisance parameters with/without clutter interference are studied. The strategy adopted in this correspondence is the use of the hypothesis test formulation (more precisely, the generalized likelihood ratio test (GLRT)). This choice is motivated by the nice property of the GLRT (i.e., it is an asymptotically uniformly most powerful (UMP) test among all the invariant statistical tests [15], which is the strongest statement of optimality that one could expect to obtain). Furthermore, in this work, it is shown that the proposed test has the same behavior compared to the (ideal) clairvoyant detector in the Neyman–Pearson sense.

Consequently, in this paper, we derive closed form expressions of the SRL in known/unknown parameters of interest and known/unknown nuisance parameters. Finally, theoretical and numerical analysis of the SRL are given for several scenarios.

II. PROBLEM SETUP

A. Observation Model

The output of a MIMO radar with widely spaced arrays where M targets are present is modelled for the ℓth pulse as follows [4]:

$$\boldsymbol{X}_\ell = \sum_{m=1}^{M} \rho_m e^{2i\pi f_m \ell} \boldsymbol{a}_{\mathcal{R}}\left(\omega_m^{(\mathcal{R})}\right) \boldsymbol{a}_{\mathcal{T}}\left(\omega_m^{(\mathcal{T})}\right)^T \boldsymbol{S}$$
$$+ \boldsymbol{W}_\ell, \ \ell \in [0 : L-1]$$

where L, ρ_m and f_m denote the number of samples per pulse period, a complex coefficient proportional to the radar cross section (RCS) and the normalized Doppler frequency of the mth target, respectively. Let T, $N_{\mathcal{T}}$ and $N_{\mathcal{R}}$ denote the number of snapshots, the number of sensors at the transmitter and the receiver, respectively.[1] Then, the known $N_{\mathcal{T}} \times T$ signal source matrix is defined by $\boldsymbol{S} = [\boldsymbol{s}_0 \ \dots \ \boldsymbol{s}_{N_{\mathcal{T}}-1}]^T$ where $\boldsymbol{s}_{N_t} = [s_{N_t}(1) \ \dots \ s_{N_t}(T)]^T$, $N_t \in \{0, \dots, N_{\mathcal{T}} - 1\}$, whereas, the $N_{\mathcal{R}} \times T$ noise matrix for the ℓth pulse is denoted $\boldsymbol{W}_\ell$. The transmitter steering and receiver steering vectors are denoted $\boldsymbol{a}_{\mathcal{T}}(\cdot)$ and $\boldsymbol{a}_{\mathcal{R}}(\cdot)$. The ith elements of these steering vectors are given by $[\boldsymbol{a}_{\mathcal{T}}(\omega_m^{(\mathcal{T})})]_i = e^{j\omega_m^{(\mathcal{T})}d_i^{(\mathcal{T})}}$ and $[\boldsymbol{a}_R(\omega_m^{(\mathcal{R})})]_i = e^{j\omega_m^{(\mathcal{R})}d_i^{(\mathcal{R})}}$ where $\omega_m^{(\mathcal{T})} = \frac{2\pi}{\nu} \sin(\psi_m)$ and $\omega_m^{(\mathcal{R})} = \frac{2\pi}{\nu} \sin(\theta_m)$ in which ψ_m is the angle of the target with respect to the transmit array (i.e., direction of departures (DOD)), where θ_m is the angle of the target with respect to the reception array (i.e., direction of arrivals (DOA)), and where ν

[1]In the following, the upper/subscript calligraphic letters $\mathcal{T}$ and $\mathcal{R}$ denote the transmitter and the receiver part, respectively.

is the wavelength. The distance between a reference sensors (the first sensor herein) and the ith sensor is denoted by $d_i^{(T)}$ and $d_i^{(R)}$ for the transmission and the reception arrays, respectively.[2]

The diversity of the MIMO radar in terms of waveform coding allows to transmit orthogonal waveforms [2], such that, $SS^H = S^*S^T = TI_{N_T}$. After matched filtering [16], one obtains

$$
\begin{aligned}
Y_\ell &= \frac{1}{\sqrt{T}} X_\ell S^H \\
&= \sum_{m=1}^{M} \alpha_m e^{2i\pi f_m \ell} a_{\mathcal{R}}\left(\omega_m^{(\mathcal{R})}\right) a_{\mathcal{T}}\left(\omega_m^{(\mathcal{T})}\right)^T + Z_\ell
\end{aligned}
$$

where $\alpha_m = \sqrt{T}\rho_m$ and where $Z_\ell = \frac{1}{\sqrt{T}} W_\ell S^H$ denotes the noise matrix after the matched filtering. It is straightforward to rewrite the above matrix-based expression as a vectorized CanDecomp/Parafac [17], [18] model of dimension $P = 3$ according to $y = [\text{vec}(Y_0)^T \dots \text{vec}(Y_{L-1})^T]^T = x + z$, where vec denotes the *vectorization* operator, $z = [z_0^T \dots z_{L-1}^T]^T$ with $z_\ell = \text{vec}(Z_\ell)$ and

$$
x = \sum_{m=1}^{M} \alpha_m g_m \tag{1}
$$

in which $c(f_m) = [1\ e^{2i\pi f_m} \dots e^{2i\pi f_m(L-1)}]^T$, $g_m = (c(f_m) \otimes a_{\mathcal{T}}(\omega_m^{(\mathcal{T})}) \otimes a_{\mathcal{R}}(\omega_m^{(\mathcal{R})}))$ and where $\otimes$ denotes the Kronecker product.

B. Statistic of the Observations

Assuming that the noise interferences (before the matched filtering) are complex circular Gaussian independent and identically distributed samples with zero-mean and a covariance matrix $\sigma^2 I$ [1] and, thanks to the waveforms orthogonality, one can notice that $E(z_\ell z_\ell^H) = \frac{1}{T}(S^* \otimes I_{N_{\mathcal{R}}}) E(\text{vec}(W_\ell)\text{vec}(W_{\ell'})^H)(S^T \otimes I_{N_{\mathcal{R}}}) = \sigma^2 I_{N_{\mathcal{T}}N_{\mathcal{R}}}$ and that $E(z_\ell z_{\ell'}^H) = 0$ for $\ell \neq \ell'$. Thus, $E(zz^H) = \sigma^2 I_{LN_{\mathcal{T}}N_{\mathcal{R}}}$. Consequently, the observation follows a complex circular Gaussian distribution $y \sim \mathcal{CN}(x, \sigma^2 I_{LN_{\mathcal{T}}N_{\mathcal{R}}})$.

III. DETECTION APPROACH

Without loss of generality, in the remain of the paper, we consider that the targets of interest are the first and the second one. The $M-2$ remaining targets consist of the clutter interference.

A. Hypothesis Test Formulation

Resolving two closely spaced sources, with respect to their parameter of interest $\omega_m^{(T)}$ and $\omega_m^{(\mathcal{R})}$, can be formulated as a binary hypothesis test (see [7], [8], [13], [19] and references therein). The hypothesis $\mathcal{H}_0$ represents the case where the two emitted signal sources are combined into one signal, whereas the hypothesis $\mathcal{H}_1$ embodies the situation where the two signals are resolvable. Since the DOAs and the DODs are the considered parameters of interest (i.e., $\omega_m^{(T)}$ and $\omega_m^{(\mathcal{R})}$ allow us to localize the targets), thus, one obtains the following binary hypothesis test:

$$
\begin{cases}
\mathcal{H}_0: & (\delta_{\mathcal{R}}, \delta_{\mathcal{T}}) = (0,0) \\
\mathcal{H}_1: & (\delta_{\mathcal{R}}, \delta_{\mathcal{T}}) \neq (0,0)
\end{cases} \tag{2}
$$

where the so-called Local SRLs (LSRL) are the local separations given by $\delta_{\mathcal{T}} \triangleq \omega_2^{(T)} - \omega_1^{(T)}$ and $\delta_{\mathcal{R}} \triangleq \omega_2^{(\mathcal{R})} - \omega_1^{(\mathcal{R})}$ which resolve the binary hypothesis test (2). Since the LSRLs are unknown, it is impossible to design an optimal detector in the Neyman–Pearson sense. Alternatively, the GLRT [15] is a well known approach appropriate to solve such a problem. The GLRT statistic is expressed

as $G(y) = \frac{p(y;\hat{\delta}_{\mathcal{R}},\hat{\delta}_{\mathcal{T}},\hat{\rho}_1,\mathcal{H}_1)}{p(y;\hat{\rho}_0,\mathcal{H}_0)} \gtrless_{\mathcal{H}_0}^{\mathcal{H}_1} \eta'$, in which $p(y;\hat{\rho}_0,\mathcal{H}_0)$ and $p(y;\hat{\delta}_{\mathcal{R}},\hat{\delta}_{\mathcal{T}},\hat{\rho}_1,\mathcal{H}_1)$ denote the probability density functions of the observation under $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. η', $\hat{\delta}_{\mathcal{R}}$, $\hat{\delta}_{\mathcal{T}}$ and $\hat{\rho}_i$ denote the detection threshold, the maximum likelihood estimate (MLE) of $\delta_{\mathcal{R}}$ and $\delta_{\mathcal{T}}$ under $\mathcal{H}_1$ and the MLE of the parameter vector ρ_i (containing all the unknown nuisance and/or unwanted parameters) under $\mathcal{H}_i, i = 0, 1$, respectively.

One can easily see that the derivation of $\hat{\delta}_{\mathcal{R}}$ and $\hat{\delta}_{\mathcal{T}}$ is a nonlinear optimization problem which is analytically intractable. Using the fact that the separation is small [7], [8], [13], [19], [20] (this assumption can be argued by the fact that the high resolution algorithms have asymptotically an infinite resolution power), one can approximate the model (1) into a model which is linear w.r.t. the unknown parameters.

B. Linear Form of the MIMO Model

First, let us introduce the so-called center parameters $\omega_c^{(T)} \triangleq \frac{\omega_1^{(T)}+\omega_2^{(T)}}{2}$ and $\omega_c^{(\mathcal{R})} \triangleq \frac{\omega_1^{(\mathcal{R})}+\omega_2^{(\mathcal{R})}}{2}$. Second, as in [7], [13], and [19], we use the first order Taylor expansion of (1) around $\delta_{\mathcal{T}} = 0$ and $\delta_{\mathcal{R}} = 0$, thus, one obtains $a_{\mathcal{T}}(\omega_1^{(T)}) = a_{\mathcal{T}}(\omega_c^{(T)}) - \frac{i}{2}\delta_{\mathcal{T}}\dot{a}_{\mathcal{T}}(\omega_c^{(T)})$, $a_{\mathcal{R}}(\omega_1^{(\mathcal{R})}) = a_{\mathcal{R}}(\omega_c^{(\mathcal{R})}) - \frac{i}{2}\delta_{\mathcal{R}}\dot{a}_{\mathcal{R}}(\omega_c^{(\mathcal{R})})$, $a_{\mathcal{T}}(\omega_2^{(T)}) = a_{\mathcal{T}}(\omega_c^{(T)}) + \frac{i}{2}\delta_{\mathcal{T}}\dot{a}_{\mathcal{T}}(\omega_c^{(T)})$ and $a_{\mathcal{R}}(\omega_2^{(\mathcal{R})}) = a_{\mathcal{R}}(\omega_c^{(\mathcal{R})}) + \frac{i}{2}\delta_{\mathcal{R}}\dot{a}_{\mathcal{R}}(\omega_c^{(\mathcal{R})})$, where $\dot{a}_{\mathcal{T}}(\cdot) \triangleq a_{\mathcal{T}}(\cdot) \odot d_{\mathcal{T}}$, and $\dot{a}_{\mathcal{R}}(\cdot) \triangleq a_{\mathcal{R}}(\cdot) \odot d_{\mathcal{R}}$ and where $\odot$ denotes the Hadamard product, $d_T = [d_0^{(T)}\ d_1^{(T)}\ \dots\ d_{N-1}^{(T)}]^T$ and where $d_R = [d_0^{(\mathcal{R})}\ d_1^{(\mathcal{R})}\ \dots\ d_{N-1}^{(\mathcal{R})}]^T$. Thus, one can rewrite the observation model as

$$
y = \check{G}\check{\zeta} + \check{D}\check{\alpha} + z \tag{3}
$$

where the $(LN_{\mathcal{T}}N_{\mathcal{R}}) \times 4$ matrix $\check{G}$ is defined as $\check{G} = [\varrho_1\ \varrho_2\ \varrho_3\ \varrho_4]$, in which $\varrho_1 = c(f) \otimes a_{\mathcal{T}}(\omega_c^{(T)}) \otimes a_{\mathcal{R}}(\omega_c^{(\mathcal{R})})$, $\varrho_2 = c(f) \otimes a_{\mathcal{T}}(\omega_c^{(T)}) \otimes \dot{a}_{\mathcal{R}}(\omega_c^{(\mathcal{R})})$, $\varrho_3 = c(f) \otimes \dot{a}_{\mathcal{T}}(\omega_c^{(T)}) \otimes a_{\mathcal{R}}(\omega_c^{(\mathcal{R})})$, and $\varrho_4 = c(f) \otimes \dot{a}_{\mathcal{T}}(\omega_c^{(T)}) \otimes \dot{a}_{\mathcal{R}}(\omega_c^{(\mathcal{R})})$. The unknown 4×1 parameter vector is given by $\check{\zeta} = [\alpha_1 + \alpha_2\ \ \frac{i}{2}\delta_{\mathcal{R}}(\alpha_2 - \alpha_1)\ \ \frac{i}{2}\delta_{\mathcal{T}}(\alpha_2 - \alpha_1)\ \ \frac{-1}{4}\delta_{\mathcal{R}}\delta_{\mathcal{T}}(\alpha_1 + \alpha_2)]^T$, whereas $\check{D} = [g_3\ \dots\ g_M]$ and $\check{\alpha} = [\alpha_3\ \dots\ \alpha_M]^T$. Rearranging (3), one obtains

$$
y = G\zeta + D\alpha + z \tag{4}
$$

where $G = [\varrho_2\ \varrho_3\ \varrho_4]$, the clutter interference $D = [\varrho_1\ g_3\ \dots\ g_M]$, $\alpha = [\alpha_1 + \alpha_2\ \alpha_3\ \dots\ \alpha_M]^T$ and $\zeta = \frac{i}{2}[\delta_{\mathcal{R}}(\alpha_2 - \alpha_1)\ \ \delta_{\mathcal{T}}(\alpha_2 - \alpha_1)\ \ \frac{i}{2}\delta_{\mathcal{R}}\delta_{\mathcal{T}}(\alpha_1 + \alpha_2)]^T$.

C. Assumptions

Throughout the rest of the paper, the following assumptions are assumed to hold:

$\mathcal{A}1$) The parameters $\omega_c^{(T)}$ and $\omega_c^{(\mathcal{R})}$ (which represent the center parameters) are assumed to be known [8] or previously estimated [7].

$\mathcal{A}2$) For sake of simplicity the Doppler frequency f_1 and f_2 are assumed to be equal[3] to f (possibly equal to zero).

$\mathcal{A}3$) Finally, the clutter interference D is known or previously estimated [21]. However, one should note that $\alpha_m, i = 1 \dots M$ are considered as unknown unequal deterministic complex parameters.[4]

In the following, we use the linear form of the signal model (4). Both cases of known and unknown noise variance will be considered.

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[2]E.g., in the case of uniform linear transmission array (ULTA), $d_i^{(T)} = (i-1)d_{\mathcal{T}}$ where $d_{\mathcal{T}}$ is the interelement space between two successive transmission sensors

[3]Nevertheless, numerical simulations will show that the derived SRL (with equal Doppler frequency assumption) has the same behavior compared to the clairvoyant detector.

[4]One should note that the case of $f_1 \neq f_2$, the case of unknown $\omega_c^{(T)}$, the case of unknown $\omega_c^{(\mathcal{R})}$ and the case of unknown clutter interference D leads to an untractable solution of the GLRT and, consequently, is beyond the scope of this paper.

IV. Derivations of the SRL

A. Case of a Known Noise Variance

1) Case of Two Targets With Interference Clutter: We consider the case where two closely spaced targets are imbedded into clutter interference. The noise variance is assumed to be known. Consequently, using the linear form in (4), the binary hypothesis test in (2) can be reformulated as follows:

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{y} = \boldsymbol{D\alpha} + \boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{D\alpha}, \sigma^2 \boldsymbol{I}), \\ \mathcal{H}_1: & \boldsymbol{y} = \boldsymbol{G\zeta} + \boldsymbol{D\alpha} + \boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{G\zeta} + \boldsymbol{D\alpha}, \sigma^2 \boldsymbol{I}). \end{cases} \quad (5)$$

Based on (5), the unconstrained MLEs of the unknown parameters are given by [22]

$$\hat{\boldsymbol{\zeta}} = (\boldsymbol{G}^H \boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{G})^{-1} \boldsymbol{G}^H \boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{y}$$
$$\hat{\boldsymbol{\alpha}}_{\mathcal{H}_0} = (\boldsymbol{D}^H \boldsymbol{D})^{-1} \boldsymbol{D}^H \boldsymbol{y},$$
$$\hat{\boldsymbol{\alpha}}_{\mathcal{H}_1} = (\boldsymbol{D}^H \boldsymbol{P}_{\boldsymbol{G}}^\perp \boldsymbol{D})^{-1} \boldsymbol{D}^H \boldsymbol{P}_{\boldsymbol{G}}^\perp \boldsymbol{y} \quad (6)$$

where $\boldsymbol{P}_{\boldsymbol{D}}^\perp \triangleq \boldsymbol{I} - \boldsymbol{P}_{\boldsymbol{D}}$, in which $\boldsymbol{P}_{\boldsymbol{D}}$ denotes the orthogonal projector onto the subspace spanned by the columns of the matrix $\boldsymbol{D}$.

Consequently, the MLEs of the noise vector are given by (7), shown at the bottom of the page, where the oblique projectors $\boldsymbol{E}_{\boldsymbol{GD}}$ and $\boldsymbol{E}_{\boldsymbol{DG}}$ are defined as $\boldsymbol{E}_{\boldsymbol{GD}} = \boldsymbol{G}(\boldsymbol{G}^H \boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{G})^{-1} \boldsymbol{G}^H \boldsymbol{P}_{\boldsymbol{D}}^\perp$ and $\boldsymbol{E}_{\boldsymbol{DG}} = \boldsymbol{D}(\boldsymbol{D}^H \boldsymbol{P}_{\boldsymbol{G}}^\perp \boldsymbol{D})^{-1} \boldsymbol{D}^H \boldsymbol{P}_{\boldsymbol{G}}^\perp$ [23]. Now, we are ready to use the statistic $T'(\boldsymbol{y})$ of the GLRT which is defined as follows:

$$T'(\boldsymbol{y}) \triangleq 2 \ln G(\boldsymbol{y}) = \frac{2}{\sigma^2} \left(\|\hat{z}_{\mathcal{H}_0}\|^2 - \|\hat{z}_{\mathcal{H}_1}\|^2 \right)$$
$$= \frac{2}{\sigma^2} \boldsymbol{y}^H \left(\boldsymbol{P}_{\boldsymbol{D}}^\perp - \boldsymbol{P}_{[\boldsymbol{GD}]}^\perp \right) \boldsymbol{y}. \quad (8)$$

Using [24, eq. (3.7)] and ([22, eq. (19)], one has $\boldsymbol{P}_{\boldsymbol{D}}^\perp - \boldsymbol{P}_{[\boldsymbol{GD}]}^\perp = \boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{E}_{\boldsymbol{GD}} \boldsymbol{P}_{\boldsymbol{D}}^\perp = \boldsymbol{P}_{\boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{G}}$. Thus,

$$T'(\boldsymbol{y}) = \frac{2}{\sigma^2} \boldsymbol{y}^H \boldsymbol{P}_{\boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{G}} \boldsymbol{y}. \quad (9)$$

Let $\boldsymbol{P}_{\boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{G}} = \boldsymbol{U} \boldsymbol{U}^H$ be any orthogonal decomposition [25] of the projector $\boldsymbol{P}_{\boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{G}}$ such that $\boldsymbol{U}^H \boldsymbol{U} = \boldsymbol{I}$ and define an auxiliary random variable $\tilde{\boldsymbol{y}} = \boldsymbol{U}^H \boldsymbol{y}$. One should note that

$$\begin{cases} \tilde{\boldsymbol{y}} = \boldsymbol{U}^H \boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{0}, \sigma^2 \boldsymbol{I}) & \text{under } \mathcal{H}_0 \\ \tilde{\boldsymbol{y}} = \boldsymbol{U}^H \boldsymbol{G\zeta} + \boldsymbol{U}^H \boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{U}^H \boldsymbol{G\zeta}, \sigma^2 \boldsymbol{I}) & \text{under } \mathcal{H}_1. \end{cases} \quad (10)$$

Consequently,

$$T'(\boldsymbol{y}) \sim \begin{cases} \chi_{2r}^2 & \text{under } \mathcal{H}_0 \\ \chi_{2r}^2(\lambda_{\mathrm{K}}(P_{\mathrm{fa}}, P_d)) & \text{under } \mathcal{H}_1 \end{cases} \quad (11)$$

in which P_{fa} and P_d denote the probability of false alarm and the probability of detection, respectively, where the subscript K stands for the case of Known noise variance, χ_{2r}^2 and $\chi_{2r}^2(\lambda_{\mathrm{K}}(P_{\mathrm{fa}}, P_d))$ denote the central and the noncentral chi-square distribution with $2r$ degrees of freedom, respectively, in which $r = \mathrm{trace}(\boldsymbol{P}_{\boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{G}}) = \mathrm{rank}(\boldsymbol{P}_{\boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{G}}) = L N_{\mathcal{T}} N_{\mathcal{R}} - M + 1$ [26]. The noncentrality parameter is given by

$$\lambda_{\mathrm{K}}(P_{\mathrm{fa}}, P_d) = \frac{\boldsymbol{\zeta}^H \boldsymbol{G}^H \boldsymbol{U} \boldsymbol{U}^H \boldsymbol{G\zeta}}{\sigma^2/2} = \frac{2 \boldsymbol{\zeta}^H \boldsymbol{G}^H \boldsymbol{P}_{\boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{G}} \boldsymbol{G\zeta}}{\sigma^2}. \quad (12)$$

Note that $\lambda_{\mathrm{K}}(P_{\mathrm{fa}}, P_d)$ can be numerically computed as the solution of $\mathcal{Q}_{\chi_{2r}^2}^{-1}(P_{\mathrm{fa}}) = \mathcal{Q}_{\chi_{2r}^2(\lambda_{\mathrm{K}}(P_{\mathrm{fa}}, P_d))}^{-1}(P_d)$, where $\mathcal{Q}_{\chi_{2r}^2}(\cdot)$ and $\mathcal{Q}_{\chi_{2r}^2(\lambda_{\mathrm{K}}(P_{\mathrm{fa}}, P_d))}(\cdot)$ denote the right tails of the pdf χ_{2r}^2 and the pdf $\chi_{2r}^2(\lambda_{\mathrm{K}}(P_{\mathrm{fa}}, P_d))$, respectively.

Result 1: The minimum SNR required to resolve two closely spaced targets in clutter interference in the known noise variance case is given by

$$\mathrm{SNR}_{\mathrm{K}} \triangleq \frac{\mathrm{trace}\{\boldsymbol{S}\boldsymbol{S}^H\}}{T\sigma^2} = \frac{N_{\mathcal{T}} \lambda_{\mathrm{K}}(P_{\mathrm{fa}}, P_d)}{2 \boldsymbol{\zeta}^H \boldsymbol{G}^H \boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{G\zeta}} = \frac{N_{\mathcal{T}} \lambda_{\mathrm{K}}(P_{\mathrm{fa}}, P_d)}{2 \|\boldsymbol{U}^H \boldsymbol{G\zeta}\|^2}. \quad (13)$$

2) Case of Two Targets Without Interference: The case of two targets without interference can be deduced from the previous result. First, note that without interference the matrix $\boldsymbol{D}$ becomes a column vector equal to $\boldsymbol{\varrho}_1$. Second, using [22, eq. (19)], one has

$$\boldsymbol{G}^H \boldsymbol{P}_{\boldsymbol{P}_{\boldsymbol{\varrho}_1}}^\perp \boldsymbol{G} = \boldsymbol{G}^H \left(\boldsymbol{P}_{\boldsymbol{\varrho}_1}^\perp - \boldsymbol{P}_{[\boldsymbol{\varrho}_1 \boldsymbol{G}]}^\perp \right) \boldsymbol{G}$$
$$= \boldsymbol{G}^H \left(\boldsymbol{P}_{[\boldsymbol{\varrho}_1 \boldsymbol{G}]} - \frac{\boldsymbol{\varrho}_1 \boldsymbol{\varrho}_1^H}{L N_{\mathcal{T}} N_{\mathcal{R}}} \right) \boldsymbol{G}$$
$$= L \left(\boldsymbol{\Phi} - \frac{1}{N_{\mathcal{T}} N_{\mathcal{R}}} \boldsymbol{\kappa} \boldsymbol{\kappa}^T \right) \quad (14)$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} f_{0,2} & f_{1,1} & f_{1,2} \\ f_{1,1} & f_{2,0} & f_{2,1} \\ f_{1,2} & f_{2,1} & f_{2,2} \end{bmatrix},$$
$$\boldsymbol{\kappa} = \begin{bmatrix} f_{0,1} & f_{1,0} & f_{1,1} \end{bmatrix}^H,$$

in which $f_{p,q} = \sum_{n_t=1}^{N_{\mathcal{T}}} (d_{n_t}^{(\mathcal{T})})^p \sum_{n_r=1}^{N_{\mathcal{R}}} (d_{n_r}^{(\mathcal{R})})^q$. By denoting

$$\boldsymbol{K} \triangleq \frac{1}{N_{\mathcal{T}}} (\boldsymbol{\Phi} - \frac{1}{N_{\mathcal{T}} N_{\mathcal{R}}} \boldsymbol{\kappa} \boldsymbol{\kappa}^T)$$

and by plugging (14) into (13), one obtains the following result.

Result 2: The relationship between the SRL ($\delta_{\mathcal{T}}$ and $\delta_{\mathcal{R}}$) and the minimum SNR, required to resolve two closely spaced sources, is then given by

$$\mathrm{SNR}_{\mathrm{K}} = \frac{\lambda_{\mathrm{K}}(P_{\mathrm{fa}}, P_d)}{2 L \boldsymbol{\zeta}^H \boldsymbol{K} \boldsymbol{\zeta}}. \quad (15)$$

3) Case of Two Targets Without Interference and With Symmetric Arrays: By symmetric arrays we mean $f_{p,1} = f_{1,p} = 0$, $\forall p$. The expression of the minimum SNR (required to resolve two closely spaced targets) becomes more compact as follows.

Result 3: The relationship between the SRL ($\delta_{\mathcal{T}}$ and $\delta_{\mathcal{R}}$) and the minimum SNR, required to resolve two closely spaced targets, for symmetric arrays is then given by $\mathrm{SNR}_{\mathrm{K}_{\mathrm{sym}}} = \frac{2 N_{\mathcal{T}} \lambda_{\mathrm{K}}(P_{\mathrm{fa}}, P_d)}{L((\delta_{\mathcal{R}}^2 + \delta_{\mathcal{T}}^2)(\alpha_2 - \alpha_1)^2 + \delta_{\mathcal{R}}^2 \delta_{\mathcal{T}}^2 (\alpha_2 + \alpha_1)^2)}$.

B. Case of Unknown Noise Variance

1) Case of Two Targets With Interference Clutter: One can extend the latter analysis to the case of unknown noise variance σ^2. The observations under each hypothesis are given by (16), shown at the bottom of the next page.

$$\begin{cases} \hat{z}_{\mathcal{H}_0} = \boldsymbol{y} - \boldsymbol{D}\hat{\boldsymbol{\alpha}} = \boldsymbol{P}_{\boldsymbol{D}}^\perp \boldsymbol{y} & \text{under } \mathcal{H}_0, \\ \hat{z}_{\mathcal{H}_1} = \boldsymbol{y} - \boldsymbol{G}\hat{\boldsymbol{\zeta}} - \boldsymbol{D}\hat{\boldsymbol{\alpha}} = (\boldsymbol{I} - \boldsymbol{E}_{\boldsymbol{GD}} - \boldsymbol{E}_{\boldsymbol{DG}}) \boldsymbol{y} = \boldsymbol{P}_{[\boldsymbol{GD}]}^\perp \boldsymbol{y} & \text{under } \mathcal{H}_1, \end{cases} \quad (7)$$

Consequently, from (16), the GLRT is given by

$$G(\boldsymbol{y}) = \frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} = \frac{||\hat{z}_{\mathcal{H}_0}||^2}{||\hat{z}_{\mathcal{H}_1}||^2} \qquad (17)$$

where the MLE of the noise variance under each hypothesis is given by [27]

$$\hat{\sigma}_i^2 = \frac{1}{NL}||\hat{z}_{\mathcal{H}_i}||^2, \quad i = 0,1. \qquad (18)$$

After some straightforward derivations, one obtains (19), shown at the bottom of the page, where $\hat{\boldsymbol{\zeta}}$, $\hat{\boldsymbol{\alpha}}_{\mathcal{H}_0}$, and $\hat{\boldsymbol{\alpha}}_{\mathcal{H}_1}$ are given by (6), respectively. In this case, it is more convenient to define the statistic $T''(\boldsymbol{y})$ as follows:

$$T''(\boldsymbol{y}) \triangleq (\ln G(\boldsymbol{y}))^{\frac{1}{NL}} - 1 = \frac{T'(\boldsymbol{y})}{N(\boldsymbol{y})} \qquad (20)$$

where $N(\boldsymbol{y}) = \frac{2}{\sigma^2} \boldsymbol{y}^H \boldsymbol{P}_{[\boldsymbol{GD}]}^\perp \boldsymbol{y}$. In addition, using any orthogonal decomposition [25], one has $\boldsymbol{P}_{[\boldsymbol{GD}]}^\perp = \boldsymbol{U}' \boldsymbol{U}'^H$. Consequently, $N(\boldsymbol{y}) = \frac{1}{\sigma^2}||\bar{\boldsymbol{y}}||^2$, in which $\bar{\boldsymbol{y}} = \boldsymbol{U}'^H \boldsymbol{y}$. Thus [see (21), shown at the bottom of the page], where $r' = \mathrm{trace}(\boldsymbol{P}_{[\boldsymbol{GD}]}^\perp) = \mathrm{rank}(\boldsymbol{P}_{[\boldsymbol{GD}]}) = LN_{\mathcal{T}}N_{\mathcal{R}} - (M+2)$.

Furthermore, one can notice that the random variables $||\bar{\boldsymbol{y}}||^2$ and $||\tilde{\boldsymbol{y}}||^2$ are independent.[5] Consequently, a new statistic $V(\boldsymbol{y})$ can be introduced as follows

$$V(\boldsymbol{y}) \triangleq \frac{r'}{r} T''(\boldsymbol{y}) \sim \begin{cases} F_{2r,2r'} & \text{under} \quad \mathcal{H}_0 \\ F_{2r,2r'}(\lambda_{\mathrm{U}}(P_{\mathrm{fa}}, P_d)) & \text{under} \quad \mathcal{H}_1 \end{cases} \qquad (22)$$

[5]Since $E(\bar{\boldsymbol{y}}) = 0$ under $\mathcal{H}_0$ and $\mathcal{H}_1$, one has

$$\begin{aligned} \mathrm{Cov}(\bar{\boldsymbol{y}}, \bar{\boldsymbol{y}}) &= E(\bar{\boldsymbol{y}}\bar{\boldsymbol{y}}^H) = \boldsymbol{U}'^H E(\boldsymbol{y}\boldsymbol{y}^H) \boldsymbol{U} \\ &= \boldsymbol{U}'^H \boldsymbol{U}' \boldsymbol{U}'^H E(\boldsymbol{y}\boldsymbol{y}^H) \boldsymbol{U} \boldsymbol{U}^H \boldsymbol{U} \\ &= \boldsymbol{U}'^H \boldsymbol{P}_{[\boldsymbol{GD}]}^\perp E(\boldsymbol{y}\boldsymbol{y}^H) \boldsymbol{P}_{\boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{G}} \boldsymbol{U} \\ &= \boldsymbol{U}'^H (\sigma^2 \boldsymbol{P}_{[\boldsymbol{GD}]}^\perp \boldsymbol{P}_{\boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{G}} + (\boldsymbol{P}_{[\boldsymbol{GD}]}^\perp \boldsymbol{e})(\boldsymbol{P}_{\boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{G}} \boldsymbol{e})^H) \boldsymbol{U} \end{aligned}$$

where $\boldsymbol{e} = \boldsymbol{G}\boldsymbol{\zeta} + \boldsymbol{D}\boldsymbol{\alpha}$ under $\mathcal{H}_1$ and $\boldsymbol{e} = \boldsymbol{D}\boldsymbol{\alpha}$ under $\mathcal{H}_0$. Noting that $\boldsymbol{P}_{[\boldsymbol{GD}]}^\perp \boldsymbol{e} = 0$, and $\boldsymbol{P}_{[\boldsymbol{BD}]}^\perp \boldsymbol{P}_{\boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{B}} = \boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp (\boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{E}_{\boldsymbol{BD}} - \boldsymbol{E}_{\boldsymbol{BD}} \boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{E}_{\boldsymbol{BD}}) \boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp = (\boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{E}_{\boldsymbol{BD}} - \boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{E}_{\boldsymbol{BD}} \boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{E}_{\boldsymbol{BD}}) \boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp = (\boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{E}_{\boldsymbol{BD}} - \boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{E}_{\boldsymbol{BD}}) \boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp = 0$. Consequently, $\mathrm{Cov}(\bar{\boldsymbol{y}}, \tilde{\boldsymbol{y}}) = 0$. Meaning that $\bar{\boldsymbol{y}}$ and $\tilde{\boldsymbol{y}}$ are uncorrelated. Thus, they are independent in the normal distribution case. Consequently, it is straightforward to conclude that $||\bar{\boldsymbol{y}}||^2$ and $||\tilde{\boldsymbol{y}}||^2$ are also independent.

where $F_{2r,2r'}$ and $F_{2r,2r'}(\lambda_{\mathrm{U}}(P_{\mathrm{fa}}, P_d))$ denote the F central and noncentral distributions [15], respectively, with $2r$ and $2r'$ degrees of freedom, in which the noncentrality parameter is given by

$$\lambda_{\mathrm{U}}(P_{\mathrm{fa}}, P_d) = \frac{2\boldsymbol{\zeta}^H \boldsymbol{G}^H \boldsymbol{P}_{\boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{G}} \boldsymbol{G}\boldsymbol{\zeta}}{\sigma^2}. \qquad (23)$$

Once again, note that the noncentrality parameter $\lambda_{\mathrm{U}}(P_{\mathrm{fa}}, P_d)$ can be computed numerically as the solution of $\mathcal{Q}_{F_{2r,2r'}}^{-1}(P_{\mathrm{fa}}) = \mathcal{Q}_{F_{2r,2r'}(\lambda_{\mathrm{U}}(P_{\mathrm{fa}}, P_d))}^{-1}(P_d)$, where $\mathcal{Q}_{F_{2r,2r'}}^{-1}(\cdot)$ and $\mathcal{Q}_{F_{2r,2r'}(\lambda_{\mathrm{U}}(P_{\mathrm{fa}}, P_d))}^{-1}(\cdot)$ denote the right tails of the pdf $F_{2r,2r'}$ and $F_{2r,2r'}(\lambda_{\mathrm{U}}(P_{\mathrm{fa}}, P_d))$, respectively.

Result 4: The SNR threshold with respect to the SRL ($\delta_{\mathcal{T}}$ and $\delta_{\mathcal{R}}$) required to resolve two closely spaced targets in the presence of clutter interference and with unknown noise variance, is given by

$$\mathrm{SNR}_{\mathrm{U}} = \frac{N_{\mathcal{T}} \lambda_{\mathrm{U}}(P_{\mathrm{fa}}, P_d)}{2||\boldsymbol{U}^H \boldsymbol{G}\boldsymbol{\zeta}||^2}. \qquad (24)$$

2) Case of Two Targets Without Interference: The case of two targets without interference can be deduced from the previous result. Using the same steps as in Subsection A.2, one obtains the following result.

Result 5: The relationship between the SRL ($\delta_{\mathcal{T}}$ and $\delta_{\mathcal{R}}$) and the minimum SNR, required to resolve two closely spaced targets with unknown noise variance, is then given by

$$\mathrm{SNR}_{\mathrm{U}} = \frac{\lambda_{\mathrm{U}}(P_{\mathrm{fa}}, P_d)}{2L\boldsymbol{\zeta}^H \boldsymbol{K}\boldsymbol{\zeta}}. \qquad (25)$$

3) Case of Two Targets Without Interference and With Symmetric Arrays: Once again, since $f_{p,1} = f_{1,p} = 0$, $\forall p$, for symmetric arrays, one has the following result.

Result 6: The relationship between the SRL ($\delta_{\mathcal{T}}$ and $\delta_{\mathcal{R}}$) and the minimum SNR, required to resolve two closely spaced targets with unknown noise variance and for symmetric arrays is given by $\mathrm{SNR}_{\mathrm{U}_{\mathrm{sym}}} = \frac{2N_{\mathcal{T}} \lambda_{\mathrm{U}}(P_{\mathrm{fa}}, P_d)}{L((\delta_{\mathcal{R}}^2 + \delta_{\mathcal{T}}^2)(\alpha_2 - \alpha_1)^2 + \delta_{\mathcal{R}}^2 \delta_{\mathcal{T}}^2(\alpha_2 + \alpha_1)^2)}$.

C. The Ideal (Clairvoyant) Detector

In the previous results, we have derived the SRL using the GLRT because the Neyman–Pearson test cannot be conducted due to the fact that $\boldsymbol{\delta}$ is an unknown parameter. Thus, it is interesting to compare $\mathrm{SNR}_{\mathrm{K}}$ and $\mathrm{SNR}_{\mathrm{U}}$ with the SNR associated

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$$\begin{cases} \mathcal{H}_0: & \boldsymbol{y} = \boldsymbol{D}\boldsymbol{\alpha} + \boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{D}\boldsymbol{\alpha}, \sigma^2 \boldsymbol{I}), \quad \sigma^2 > 0 \\ \mathcal{H}_1: & \boldsymbol{y} = \boldsymbol{G}\boldsymbol{\zeta} + \boldsymbol{D}\boldsymbol{\alpha} + \boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{G}\boldsymbol{\zeta} + \boldsymbol{D}\boldsymbol{\alpha}, \sigma^2 \boldsymbol{I}), \quad \sigma^2 > 0. \end{cases} \qquad (16)$$

$$\begin{cases} \hat{z}_{\mathcal{H}_0} = \boldsymbol{y} - \boldsymbol{D}\hat{\boldsymbol{\alpha}}_{\mathcal{H}_0} = \boldsymbol{P}_{\bar{\boldsymbol{D}}}^\perp \boldsymbol{y} & \text{under } \mathcal{H}_0 \\ \hat{z}_{\mathcal{H}_1} = \boldsymbol{y} - \boldsymbol{G}\hat{\boldsymbol{\zeta}}_{\mathcal{H}_1} - \boldsymbol{D}\hat{\boldsymbol{\alpha}}_{\mathcal{H}_1} = (\boldsymbol{I} - \boldsymbol{E}_{\boldsymbol{GD}} - \boldsymbol{E}_{\boldsymbol{DG}})\boldsymbol{y} = \boldsymbol{P}_{[\boldsymbol{GD}]}^\perp \boldsymbol{y} & \text{under } \mathcal{H}_1 \end{cases} \qquad (19)$$

$$T''(\boldsymbol{y}) = \frac{||\tilde{\boldsymbol{y}}||^2}{||\bar{\boldsymbol{y}}||^2} \qquad (21)$$

and

$$\begin{cases} \bar{\boldsymbol{y}} = \boldsymbol{U}'^H \boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{0}, \sigma^2 \boldsymbol{I}) & \text{under } \mathcal{H}_0 \\ \bar{\boldsymbol{y}} = \boldsymbol{U}'^H \boldsymbol{z} \sim \mathcal{CN}(\boldsymbol{0}, \sigma^2 \boldsymbol{I}) & \text{under } \mathcal{H}_1, \end{cases} \Rightarrow \begin{cases} N(\boldsymbol{y}) \sim \chi_{2r'}^2 & \text{under } \mathcal{H}_0 \\ N(\boldsymbol{y}) \sim \chi_{2r'}^2(0) & \text{under } \mathcal{H}_1 \end{cases}$$

to the clairvoyant Neyman–Pearson test (where all the parameter are known, i.e., α_m, $m = 1 \ldots M$ and even $\delta_\mathcal{T}$ and $\delta_\mathcal{R}$). Toward this aim, one can consider the new observation vector $\boldsymbol{y}' \triangleq \boldsymbol{y} - (\alpha_1 + \alpha_2)\boldsymbol{c}(f) \otimes \boldsymbol{a}_\mathcal{T}(\omega_c^{(\mathcal{T})}) \otimes \boldsymbol{a}_\mathcal{R}(\omega_c^{(\mathcal{R})})$. Thus, it can be shown that $\boldsymbol{y}' = \boldsymbol{G}\boldsymbol{P}^T\boldsymbol{P}\boldsymbol{\zeta} + z$, where $\boldsymbol{P} = [\boldsymbol{0}\ \boldsymbol{I}_3]$ leading to the following binary hypothesis test:

$$\begin{cases} \mathcal{H}_0: & \boldsymbol{y}' = z, \\ \mathcal{H}_1: & \boldsymbol{y}' = \boldsymbol{G}\boldsymbol{P}^T\boldsymbol{\zeta} + z. \end{cases}$$

The latter hypothesis test is a detection problem of a known deterministic signal imbedded in a complex white Gaussian noise with known variance. This is the so-called mean-shifted Gauss–Gauss detection problem such that [15]

$$T_\mathrm{C}(\boldsymbol{y}') \sim \begin{cases} \mathcal{H}_0: \mathcal{CN}(0, \frac{\sigma^2 \mathcal{E}}{2}) \\ \mathcal{H}_1: \mathcal{CN}(\mathcal{E}, \frac{\sigma^2 \mathcal{E}}{2}) \end{cases}$$

where the subscript C stands for the Clairvoyant case, and where $\mathcal{E} = \boldsymbol{\zeta}^H \boldsymbol{P}\boldsymbol{G}^H\boldsymbol{G}\boldsymbol{P}^T\boldsymbol{\zeta} = \boldsymbol{\zeta}^H \boldsymbol{\Phi}\boldsymbol{\zeta}$. On the other hand, the detection performance are given by $\lambda_\mathrm{C}(P_\mathrm{fa}, P_d) = (Q^{-1}(P_\mathrm{fa}) - Q^{-1}(P_d))^2$, in which λ_C denotes the so-called deflection coefficient, whereas $Q^{-1}(\cdot)$ is the inverse of the right tail of the probability function for a Gaussian random variable with zero mean and unit variance, whereas $\lambda_\mathrm{C}(P_\mathrm{fa}, P_d) = \frac{2\mathcal{E}}{\sigma^2}$ [15, p. 103]. Consequently, denoting $\boldsymbol{K}' = \frac{1}{N_\mathcal{T}}\boldsymbol{\Phi}$, one has the following result.

Result 7: The relationship between $\boldsymbol{\zeta}$ and the minimum SNR, required to resolve two closely spaced sources in the optimal (clairvoyant) case, is then given by

$$\mathrm{SNR}_\mathrm{C} = \frac{\lambda_\mathrm{C}(P_\mathrm{fa}, P_d)}{2L\boldsymbol{\zeta}^H \boldsymbol{K}'\boldsymbol{\zeta}}. \tag{26}$$

The next section is devoted to the theoretical and numerical analysis of the SRL (or equivalently their corresponding minimal SNRs, i.e., SNR_K, SNR_U, and SNR_C).

V. ANALYSIS OF THE SRL AND SIMULATIONS RESULTS

A. Effect of the Noise Variance's Prior on the Minimum SNR Required to Resolve Two Sources

Let us compare the derived SNR i) in the clairvoyant case, ii) in the unknown parameters with known noise variance case, and iii) and in the unknown parameters with unknown noise variance case. On one hand, from (15), (25), and (26), one obtains

$$\frac{\mathrm{SNR}_\mathrm{C}}{\mathrm{SNR}_\mathrm{K}} = \rho \frac{\lambda_\mathrm{C}(P_\mathrm{fa}, P_d)}{\lambda_\mathrm{K}(P_\mathrm{fa}, P_d)} \quad \text{where } \rho = \frac{\boldsymbol{\zeta}^H \boldsymbol{K}\boldsymbol{\zeta}}{\boldsymbol{\zeta}^H \boldsymbol{K}'\boldsymbol{\zeta}} \tag{27}$$

and

$$\frac{\mathrm{SNR}_\mathrm{K}}{\mathrm{SNR}_\mathrm{U}} = \frac{\lambda_\mathrm{K}(P_\mathrm{fa}, P_d)}{\lambda_\mathrm{U}(P_\mathrm{fa}, P_d)}. \tag{28}$$

On the other hand, note the following: $\mathcal{P}1$) for any $P_d > P_\mathrm{fa}$, one has $\lambda_\mathrm{C}(P_\mathrm{fa}, P_d) < \lambda_\mathrm{K}(P_\mathrm{fa}, P_d) < \lambda_\mathrm{U}(P_\mathrm{fa}, P_d)$ [7]; $\mathcal{P}2$) let us set $\boldsymbol{\kappa}_0 = \boldsymbol{Q}^*\boldsymbol{\kappa}/\sqrt{N_\mathcal{T}N_\mathcal{R}}$, in which $\boldsymbol{Q} = \mathrm{diag}\{\frac{j}{2}(\alpha_2 - \alpha_1), \frac{j}{2}(\alpha_2 - \alpha_1), \frac{1}{4}(\alpha_1 + \alpha_2)\}$. Then, the Hermitian matrix $\boldsymbol{\Omega} = \boldsymbol{K}' - \boldsymbol{K} = \boldsymbol{\kappa}_0\boldsymbol{\kappa}_0^H$ is a positive semi-definite matrix. Thus, $\rho \leq 1$. Consequently, from (27) and (28), $\mathcal{P}1$, and $\mathcal{P}2$, one deduces, as expected, that for fixed P_fa and P_d (such that $P_d > P_\mathrm{fa}$) one has $\mathrm{SNR}_\mathrm{C} < \mathrm{SNR}_\mathrm{K} < \mathrm{SNR}_\mathrm{U}$. In Fig. 1, we have reported the LSRL $\delta_\mathcal{R}$ in the clairvoyant, the known noise variance and the unknown noise variance cases versus the SNR (the same conclusion are done also for the LSRL $\delta_\mathcal{T}$). One can notice that the LSRLs derived in the cases of known and unknown noise variance have the same behavior as the one in the clairvoyant case. For the same SRL (i.e., for a fixed $\delta_\mathcal{T}$ and $\delta_\mathcal{R}$), the gap between
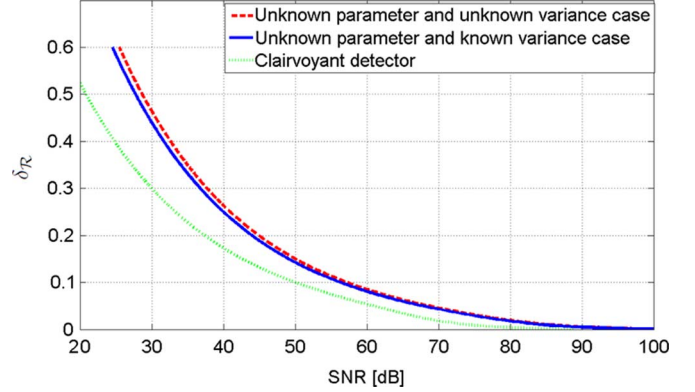

Fig. 1. The LSRL versus the required SNR to resolve two closely spaced targets for $L = 4$ samples per pulse period, ULA at the transmitter and at the receiver with $N_\mathcal{T} = N_\mathcal{R} = 4$ and $T = 100$ snapshots.

SNR_K and SNR_U is exclusively due to the noncentrality parameters $\lambda_\mathrm{K}(P_\mathrm{fa}, P_d)$ and $\lambda_\mathrm{U}(P_\mathrm{fa}, P_d)$. This gap is approximately equal to 1 dB. Whereas, the gap between SNR_C and SNR_K is due to both: i) the ratio of the deflection coefficient $\lambda_\mathrm{C}(P_\mathrm{fa}, P_d)$ over the noncentrality parameter $\lambda_\mathrm{K}(P_\mathrm{fa}, P_d)$, and, ii) the norm of $\boldsymbol{\Omega}$, which reflects the value of ρ. This latter gap, is evaluated to 9 dB.

B. Effect of the Clutter Interference

In the following, we consider that the targets of interest (i.e., the first one and the second one) are spaced by $\delta_\mathcal{T}$ and $\delta_\mathcal{R}$, whereas the $M - 2$ remain targets are equally spaced by $\Delta_\mathcal{T}$ and $\Delta_\mathcal{R}$.

In Fig. 2, we have reported the effect of additional sources (considered as a clutter interference) on the SNR threshold (i.e., the required SNR to resolve two closely spaced targets) w.r.t. $\delta_\mathcal{R}$ (the same conclusion are done also for $\delta_\mathcal{T}$). One can distinguish two cases.

1) The first one represents the scenario where $\Delta_\mathcal{T} \gg \delta_\mathcal{T}$ and $\Delta_\mathcal{R} \gg \delta_\mathcal{R}$. In this case, one can notice that the additional sources do not affect the minimal SNR [Fig. 2 (top)]. This can be explained by the fact that the high resolution algorithms have asymptotically an infinite resolving power [1], [20].

2) The second scenario is for $\Delta_\mathcal{T} > \delta_\mathcal{T}$ and $\Delta_\mathcal{R} > \delta_\mathcal{R}$. In this case, one can notice the drastic effect of the interfering sources [Fig. 2 (bottom)]. For example, the SNR gap between $M = 2$ targets and $M = 4$ targets is evaluated around 6 dB.

Finally, in Fig. 3, we investigated the relative distance $(\frac{\Delta_\mathcal{R}}{\delta_\mathcal{R}})$ from which the interfering targets start to affect the targets of interest in the case of $N_\mathcal{R} = N_\mathcal{T} = 10, L = 9$, and $T = 300$. One can note that, in these conditions, the interference targets will have an insignificant effect on the minimum SNR required to resolve the targets of interest if the relative distance between the interfering targets and the targets of interest is greater than ten (i.e., if $\Delta_\mathcal{R} > 10\,\delta_\mathcal{R}$). The same conclusion is done for $\delta_\mathcal{T}$.

VI. CONCLUSION

In this paper, we have derived the statistical resolution limit (SRL) for two closely spaced targets using a MIMO radar with widely separated arrays (made from possibly nonuniform transmitter and receiver arrays) in the presence of clutter interference. Toward this goal, we have conducted a hypothesis test approach based on the generalized likelihood ratio test (GLRT). This analysis provides useful information concerning the behavior of the SRL and the minimum SNR required to resolve two closely spaced targets for a given probability of false
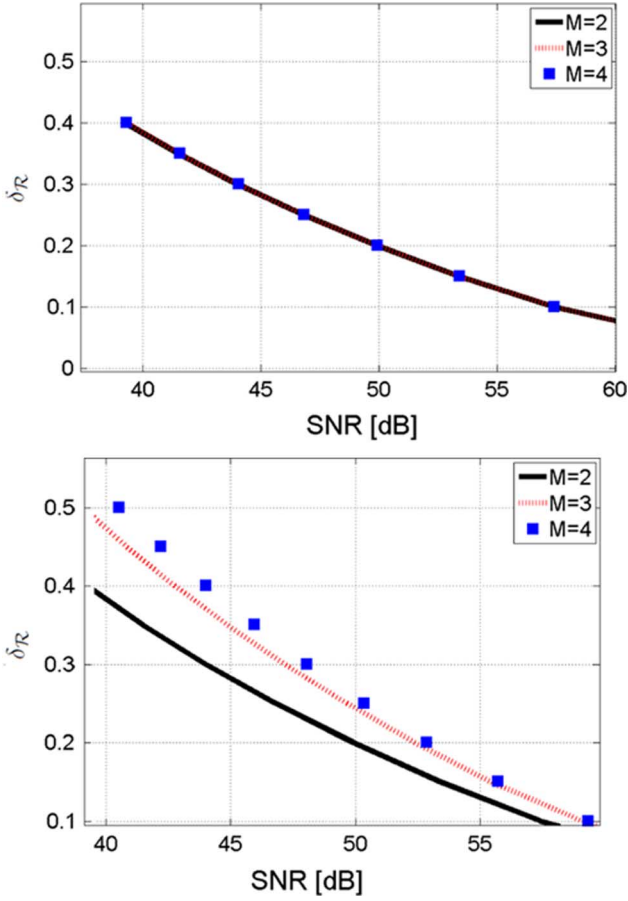
Fig. 2. Top: The minimum SNR to resolve two closely spaced targets with known noise variance for different values of M in which $\Delta_{\mathcal{R}} = \Delta_{\mathcal{T}} = 1.5$, $N_{\mathcal{T}} = N_{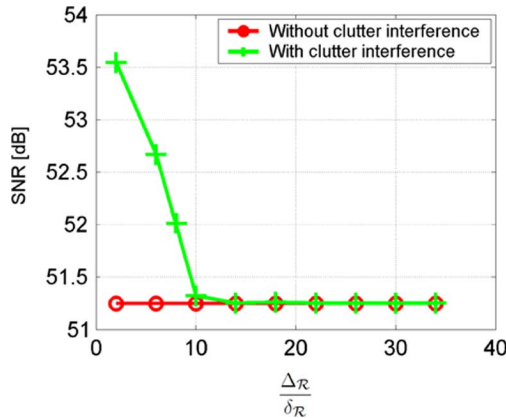\mathcal{R}} = 4$, and $T = 100$ snapshots. Bottom: The minimum SNR to resolve two closely targets with known noise variance for different values of M in which $\Delta_{\mathcal{R}} = \Delta_{\mathcal{T}} = 0.7$, $N_{\mathcal{T}} = N_{\mathcal{R}} = 4$, and $T = 100$ snapshots.

Fig. 3. The effect of the interference targets on the minimum SNR required to resolve two closely spaced targets with known noise variance versus the relative distance $\frac{\Delta_{\mathcal{R}}}{\delta_{\mathcal{R}}}$ in which $N_{\mathcal{T}} = N_{\mathcal{R}} = 10$ and $T = 300$ snapshots.

alarm and a given probability of detection. Finally, numerical simulations shows that the derived SRL has the same behavior compared to the clairvoyant (ideal) detector.

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