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Statistical analysis of achievable resolution limit in the near field source localization context

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ABSTRACT

In this fast communication, we derive the statistical resolution limit (SRL), characterizing the minimal parameter separation, to resolve two closely spaced known near-field sources impinging on a linear array. Toward this goal, we conduct on the first-order Taylor expansion of the observation model a Generalized Likelihood Ratio Test (GLRT) based on a Constrained Maximum Likelihood Estimator (CMLE) of the SRL. More precisely, the minimum separation between two near-field sources, that is detectable for a given probability of false alarm and a given probability of detection, is derived herein. Finally, numerical simulations are done to quantify the impact of the array geometry of the signal sources power distribution and of the array aperture on the statistical resolution limit.

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1. Introduction

Passive sources localization by an array of sensors is an important topic with a large number of applications, such as sonar, seismology, digital communications, etc. One can find many estimation schemes adapted to the so-called near-field source localization (e.g., [1–5]). However, to the best of our knowledge, no work has been done on the resolvability of closely spaced near-field sources.

A common tool to characterize the resolvability between two closely spaced signals is the so-called Statistical Resolution Limit (SRL). The SRL [6–12], defined as the minimal separation between two signals in terms of parameters of interest which allows a correct resolvability, is a challenging problem and an essential tool to quantify estimators performance.

The idea herein is to use the detection theory in order to derive/link the SRL to the probability of false alarm, P_{fa}

and to the probability of detection P_d . In this spirit Sharman and Milanfar [9] have studied the problem of distinguishing whether the observed signal contains one or two frequencies at a given SNR using the Generalized Likelihood Ratio Test (GLRT). In Liu and Nehorai [11], defined a statistical angular resolution limit using the asymptotic equivalence of the GLRT (in terms of snapshots). Recently, Amar and Weiss [12] proposed to determine the SRL of complex sinusoids with nearby frequencies using the Bayesian approach for a given correct decision probability.

It is important to note that all the references listed before have been conducted in the spectral analysis context or for the far-field source localization problem. To the best of our knowledge, no study/result is available concerning the near-field source localization problem. The goal of this paper is to fill this lack. More precisely, we consider the context of deriving the SRL for two complex narrow-band closely spaced near-field sources using a binary hypothesis test approach. Since the separation term is an unknown parameter, it is impossible to design an optimal detector in the Neyman–Pearson sense [13,14]. Consequently, the GLRT is applied herein. The choice of the hypothesis test strategy is motivated by the following arguments: (1) the SRL based on detection

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theory takes into account the statistical coupling between parameters (unlike the Lee criterion [6]), (2) there exists a relationship between the SRL based on the Smith criterion [7] and the SRL based on detection theory in the asymptotic case [11] (in terms of snapshots). Taking advantage of this relationship, one could deduce the SRL in the Smith sense, and finally, (3) unlike the Bayesian approach [12], the use of the GLRT does not require any *prior* knowledge on the unknown parameter of interest [13]. Consequently, in this paper we derive the SRL for two closely spaced near-field sources that allows a correct sources resolvability for an arbitrary linear array following the GLRT strategy.

2. Problem setup and assumptions

Let us consider a received signal composed of two emitted near-field and narrow-band sources impinging on a linear array (possibly nonuniform) of N sensors. The observation model is given by [3,4,15]

$$y_n(t) = \sum_{m=1}^2 s_m(t) e^{j\tau_{nm}} + v_n(t), \quad (1)$$

$t = 1, \dots, L, n = 0, \dots, N-1$ where $y_n(t)$ and $v_n(t)$ denote the noisy observed signal and the additive noise at the output of the n th sensor, respectively, whereas, $s_m(t)$ denoted the m th deterministic source signal. The number of snapshots is denoted by L and τ_{nm} is the delay associated with the signal propagation time from the first sensor to the n th sensor w.r.t. the m th source which is given by [4]

$$\tau_{nm} = \frac{2\pi r_m}{v} \left(\sqrt{1 + \frac{d_n^2}{r_m^2} - \frac{2d_n \sin \theta_m}{r_m}} - 1 \right) \quad (2)$$

where v , r_m and $\theta_m \in [0, \pi/2]$ denote the signal wavelength, the range and the bearing of the m th source, respectively. The distance between a reference sensor (the first sensor herein) and the n th sensor is denoted by d_n (e.g., in the case of Uniform Linear Array (ULA), $d_n = nd$ where d is the inter-element space between two successive sensors). It is well known that, if the source range is inside of the so-called Fresnel region [4,16], i.e.

$$0.62(D^3 v)^{1/2} < r_m < 2D^2 \frac{(N-1)^2}{v}, \quad (3)$$

where D is the array aperture, then the delay τ_{nm} can be approximated by

$$\tau_{nm} = \rho_m d_n + \kappa_m d_n^2 + o\left(\frac{d_n^2}{r_m^2}\right), \quad (4)$$

in which $\rho_m = (-2\pi/v) \sin(\theta_m)$ and $\kappa_m = (\pi/v r_m) \cos^2(\theta_m)$ denote the parameters of interest. Neglecting the term $o(d_n^2/r_m^2)$, the observation model becomes

$$y_n(t) = \sum_{m=1}^2 s_m(t) e^{j(\rho_m d_n + \kappa_m d_n^2)} + v_n(t). \quad (5)$$

Consequently, the observation vector can be expressed as

$$\mathbf{y}(t) = [y_0(t) \dots y_{N-1}(t)]^T \quad (6)$$

$$= [\mathbf{a}(\rho_1, \kappa_1) \mathbf{a}(\rho_2, \kappa_2)] \mathbf{s}(t) + \mathbf{v}(t), \quad (7)$$

in which $\mathbf{v}(t) = [v_0(t) \dots v_{N-1}(t)]^T$, $\mathbf{s}(t) = [s_1(t) s_2(t)]^T$ and where

$$[\mathbf{a}(\rho_m, \kappa_m)]_{n+1} = e^{j(\rho_m d_n + \kappa_m d_n^2)}. \quad (8)$$

Finally, the full observation vector can be written as

$$\mathbf{y} \triangleq [\mathbf{y}^T(1) \mathbf{y}^T(2) \dots \mathbf{y}^T(L)]^T. \quad (9)$$

Throughout the rest of the paper, the following assumptions are assumed to hold:

- **A1.** The additive noise is assumed to be a complex circular white Gaussian random process (uncorrelated both temporally and spatially) with zero-mean and known [9] or previously estimated [17] variance σ^2 .
- **A2.** The parameters $\rho_c = ((\rho_1 + \rho_2)/2)$ and $\kappa_c = ((\kappa_1 + \kappa_2)/2)$ (which represent the center parameters) are assumed to be known [11] or previously estimated [9]. However, in the following we prove that this assumption does not affect the SRL (since the SRL is independent of ρ_c and κ_c).
- **A3.** The sources and the array geometry are known [18–21] (the case of unknown signal sources leads to an intractable solution of the SRL and is beyond the scope of this paper).

3. Near-field statistical resolution limit

3.1. Hypothesis test formulation

In the following, we conduct a binary hypothesis test formulation to derive the SRL. Let the hypothesis \mathcal{H}_0 represents the case where the two signal sources combine into one single signal (i.e., it represents the case of two unresolvable targets), whereas the hypothesis \mathcal{H}_1 embodies the situation where the two signals are resolvable [9,11,12]. Then, the hypothesis test is given by

$$\begin{cases} \mathcal{H}_0 : \delta \triangleq \mathbf{0}, \\ \mathcal{H}_1 : \delta \neq \mathbf{0}, \end{cases} \quad (10)$$

where $\delta \triangleq [\delta_\rho \delta_\kappa]^T$ denotes the vector separation in which $\delta_\rho = \rho_2 - \rho_1$ and $\delta_\kappa = \kappa_2 - \kappa_1$. The SRL $\bar{\delta} \triangleq [\bar{\delta}_\rho \bar{\delta}_\kappa]^T$ represents the vector separation which resolves (10) for a given P_{fa} and a given P_d . The Generalized Likelihood Ratio Test (GLRT) is a well known approach to solve a composite binary hypothesis test [14]. It is expressed as follows

$$G(\mathbf{y}) = \max_{\delta_\rho, \delta_\kappa} \frac{p(\mathbf{y}; \delta_\rho, \delta_\kappa, \mathcal{H}_1)}{p(\mathbf{y}; \mathcal{H}_0)} = \frac{p(\mathbf{y}; \hat{\delta}_\rho, \hat{\delta}_\kappa, \mathcal{H}_1)}{p(\mathbf{y}; \mathcal{H}_0)} \stackrel{\mathcal{H}_1}{\geq} \gamma, \quad (11)$$

in which $p(\mathbf{y}; \cdot)$ denotes the pdf of $\mathbf{y} \sim \mathcal{CN}(\mathbf{E}(\mathbf{y}), \sigma^2 \mathbf{I})$, $\mathbf{s}_i = [s_i(1) \dots s_i(L)]^T$ for $i = 1, 2$, and where γ , $\hat{\delta}_\rho$ and $\hat{\delta}_\kappa$ denote the detection threshold, the Maximum Likelihood Estimate (MLE) under \mathcal{H}_1 of δ_ρ and δ_κ , respectively. One can note that the difficult task to derive the GLRT is to obtain an analytical expressions of $\hat{\delta}_\rho$ and $\hat{\delta}_\kappa$ since the near-field model is highly nonlinear. The key idea to overcome this problem is to consider a small separation [9]. This assumption can be argued by the fact the high resolution algorithms have, asymptotically, an infinite resolving power [22]. Consequently, in the following, we

show that the near-field model can be linearized by considering small separation on ρ and κ .

3.2. Linearized near-field model

Using parameters ρ_c and κ_c , a first-order Taylor expansion of the observation model around $(\delta_\rho, \delta_\kappa) = (0, 0)$ leads to

$$\mathbf{y} = \mathbf{A}\mathbf{s}_+ + \mathbf{D}\delta + \mathbf{v}, \quad (12)$$

where $\mathbf{s}_+ = \mathbf{s}_1 + \mathbf{s}_2$ and $\mathbf{D} = [\mathbf{B}\mathbf{s}_- \mathbf{C}\mathbf{s}_-]$ in which $\mathbf{s}_- = \mathbf{s}_2 - \mathbf{s}_1$ ($\mathbf{s}_1 \neq \mathbf{s}_2$) and $\mathbf{s}_i = [s_i(1) \dots s_i(L)]^T$ for $i=1,2$. Denoting, $\mathbf{d} = [d_0 \ d_1 \ \dots \ d_{N-1}]^T$, \otimes the Kronecker product, \odot the Hadamard product and \mathbf{I}_L the identity matrix of dimension $L \times L$, we have

$$\mathbf{A} = \mathbf{I}_L \otimes \mathbf{a}(\rho_c, \kappa_c), \quad (13)$$

$$\mathbf{B} = \frac{j}{2} \mathbf{I}_L \otimes (\mathbf{a}(\rho_c, \kappa_c) \odot \mathbf{d}) \quad (14)$$

and

$$\mathbf{C} = \frac{j}{2} \mathbf{I}_L \otimes (\mathbf{a}(\rho_c, \kappa_c) \odot \mathbf{d} \odot \mathbf{d}). \quad (15)$$

3.3. Constrained MLE (CMLE) of the SRL

Since, ρ_c , κ_c , \mathbf{s}_1 and \mathbf{s}_2 are known, observation model (12) can be simplified according to

$$\mathbf{z} \triangleq \mathbf{y} - \mathbf{A}\mathbf{s}_+ = \mathbf{D}\delta + \mathbf{v}. \quad (16)$$

As $\delta \in \mathbb{R}^2$, one has to find the Constrained MLE (CMLE) of δ in order to use correctly the GLRT. More precisely, the constrained optimization problem can be written according to

$$\arg \max_{\delta} L(\mathbf{z}, \delta) \quad \text{subject to } \Im\{\delta\} = \mathbf{0}, \quad (17)$$

where $L(\mathbf{z}, \delta) = \ln p(\mathbf{z}; \delta, \mathcal{H}_1)$ is the log-likelihood function, $\Im\{\cdot\}$ denotes the imaginary part. The Lagrange function adapted to this problem can be defined as

$$\mathcal{L}(\delta, \boldsymbol{\vartheta}) = L(\mathbf{z}, \delta) - \frac{j\boldsymbol{\vartheta}^T}{2} (\delta - \delta^*) \quad (18)$$

$$\mathcal{L}(\delta, \boldsymbol{\vartheta}) \Rightarrow \begin{cases} \frac{\partial \mathcal{L}}{\partial \delta} = \frac{-1}{\sigma^2} \mathbf{D}^T (\mathbf{z} - \mathbf{D}\delta)^* - j\frac{\boldsymbol{\vartheta}}{2} \\ \frac{\partial \mathcal{L}}{\partial \boldsymbol{\vartheta}} = \Im\{\delta\}, \end{cases} \quad (19)$$

where $\boldsymbol{\vartheta}$ is the Lagrange multiplier. Setting $(\partial \mathcal{L} / \partial \delta)|_{\hat{\delta}_0} = \mathbf{0}$ one has

$$\hat{\delta}_0 = (\mathbf{D}^H \mathbf{D})^{-1} \left(\mathbf{D}^H \mathbf{z} - \frac{j}{2} \sigma^2 \boldsymbol{\vartheta} \right), \quad (20)$$

where

$$\mathbf{D}^H \mathbf{D} = \frac{\|\mathbf{s}_-\|^2}{4} \mathbf{F}, \quad (21)$$

in which

$$\mathbf{F} = \begin{bmatrix} f_2 & f_3 \\ f_3 & f_4 \end{bmatrix} \quad (22)$$

and

$$f_i = \sum_{n=0}^{N-1} (d_n)^i. \quad (23)$$

Thus, note that $\mathbf{D}^H \mathbf{D}$ is a real matrix. Consequently, using $(\partial \mathcal{L} / \partial \boldsymbol{\vartheta})|_{\boldsymbol{\vartheta}_0} = \mathbf{0}$ and (20), one obtains

$$\boldsymbol{\vartheta}_0 = \frac{2}{\sigma^2} \Im\{\mathbf{D}^H \mathbf{z}\}. \quad (24)$$

Plugging (24) into (20) one obtains

$$\hat{\delta} = (\mathbf{D}^H \mathbf{D})^{-1} \Re\{\mathbf{D}^H \mathbf{z}\}. \quad (25)$$

3.4. Near-field SRL derivation

In the light of the above framework, the new binary hypothesis test is given by

$$\begin{cases} \mathcal{H}_0 : \mathbf{z} = \mathbf{v}, \\ \mathcal{H}_1 : \mathbf{z} = \mathbf{D}\delta + \mathbf{v}. \end{cases} \quad (26)$$

The GLRT is then expressed as

$$G(\mathbf{z}) = \frac{p(\mathbf{z}; \hat{\delta}, \mathcal{H}_1)}{p(\mathbf{z}; \mathcal{H}_0)} = e^{((\|\mathbf{z}\|^2 / \sigma^2) - (\|\mathbf{z} - \mathbf{D}\hat{\delta}\|^2) / \sigma^2)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\geq}} \eta'. \quad (27)$$

Thus,

$$\ln G(\mathbf{z}) = \frac{1}{\sigma^2} (\mathbf{z}^H \mathbf{D} \hat{\delta} + \delta^T \mathbf{D}^H \mathbf{z} - \hat{\delta}^T \mathbf{D}^H \mathbf{D} \hat{\delta}). \quad (28)$$

Plugging (25) into (28), one obtains

$$\ln G(\mathbf{z}) = \frac{1}{\sigma^2} \Re\{\mathbf{z}^H \mathbf{D}\} (\mathbf{D}^H \mathbf{D})^{-1} \Re\{\mathbf{D}^H \mathbf{z}\}. \quad (29)$$

Let us define the new statistic

$$T(\mathbf{z}) \triangleq 2 \ln G(\mathbf{z}) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\geq}} \eta = 2 \ln \eta'. \quad (30)$$

According to the Appendix, one obtains

$$T(\mathbf{z}) \sim \begin{cases} \chi_2^2(0) = \chi_2^2 & \text{under } \mathcal{H}_0, \\ \chi_2^2(\lambda(P_{fa}, P_d)) & \text{under } \mathcal{H}_1, \end{cases} \quad (31)$$

where χ_2^2 and $\chi_2^2(\lambda(P_{fa}, P_d))$ denote the central and the non-central chi-square distribution of two degrees of freedom, respectively, in which

$$\lambda(P_{fa}, P_d) = \frac{2\|\mathbf{s}_-\|^2}{\sigma^2} \bar{\boldsymbol{\delta}}^T \mathbf{F} \bar{\boldsymbol{\delta}}. \quad (32)$$

Moreover, the probability of false alarm and the probability of detection are given by

$$P_{fa} = Q_{\chi_2^2}(\eta) \quad (33)$$

and

$$P_d = Q_{\chi_2^2(\lambda(P_{fa}, P_d))}(\eta), \quad (34)$$

where $Q_{\chi_2^2}(\eta)$ and $Q_{\chi_2^2(\lambda(P_{fa}, P_d))}(\eta)$ denote the right tail of the χ_2^2 and $\chi_2^2(\lambda(P_{fa}, P_d))$ pdf starting from η . Thus, the non-centrality parameter $\lambda(P_{fa}, P_d)$ can also be expressed as the solution of

$$Q_{\chi_2^2}^{-1}(P_{fa}) = Q_{\chi_2^2(\lambda(P_{fa}, P_d))}^{-1}(P_d), \quad (35)$$

where $Q_{\chi_2^2}^{-1}$ and $Q_{\chi_2^2(\lambda(P_{fa}, P_d))}^{-1}$ are the inverse of the right tail of the χ_2^2 and $\chi_2^2(\lambda(P_{fa}, P_d))$ pdf. Consequently, one can state the following results:

Result 1. The relationship between the SRL $\bar{\delta}$ and the minimum SNR required to resolve two closely spaced known near-field sources, is given by

$$SNR \triangleq \frac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2}{\sigma^2} = \lambda(P_{fa}, P_d) \frac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2}{2\|\mathbf{s}_-\|^2 \bar{\delta}^T \mathbf{F} \bar{\delta}} \quad (36)$$

Result 2. The relationship between the SRL $\bar{\delta}$ and the minimum SNR required to resolve two orthogonal (i.e., $\mathbf{s}_1^H \mathbf{s}_2 = 0$) closely spaced known near-field sources, is given by

$$SNR_o = \frac{\lambda(P_{fa}, P_d)}{2\bar{\delta}^T \mathbf{F} \bar{\delta}}, \quad (37)$$

since $\|\mathbf{s}_-\|^2 = \|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2$. \square

Note that SNR_o is invariant in comparison with the source powers.

4. Simulation results

Two complex near-field narrow-band sources belonging to the so-called Fresnel region are impinging on a linear

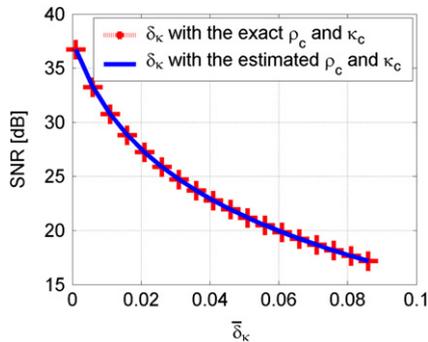


Fig. 1. The required SNR to resolve two known closely spaced near-field sources using the exact and the estimated (using the 2D-MUSIC algorithm [5]) parameters ρ_c and κ_c , for an ULA of $N=15$ sensors with $L=25$ snapshots and $\bar{\delta}_\rho = 0.02$. The same behavior is noticed w.r.t. $\bar{\delta}_\rho$.

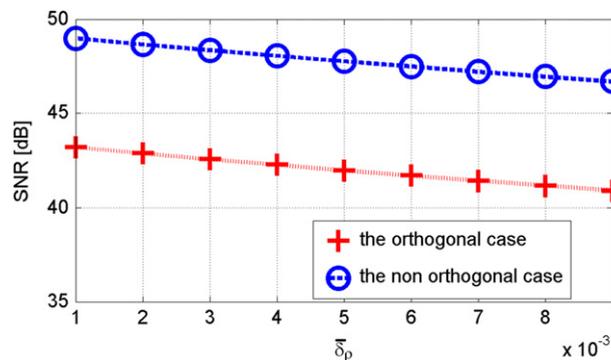


Fig. 2. The required SNR to resolve two closely spaced BPSK near-field sources in the case of real orthogonal signal sources and non orthogonal signal sources, for an ULA of $N=7$ sensors with $L=25$ snapshots and $\bar{\delta}_\kappa = 0.02$. The same behavior is noticed w.r.t. $\bar{\delta}_\kappa$.

array (the geometry is detailed of each scenarios). The probability of false alarm and the probability of detection are, for example, fixed for $P_{fa} = 0.01$ and $P_d = 0.99$.

- From Result 1, one can notice that the SRL does not depend on the parameters ρ_c and κ_c . Furthermore, from the Cramér–Rao bound point of view, one can easily prove that the CRB w.r.t. ρ_1 and ρ_2 (or, κ_1 and κ_2), for two known signal sources, depends only on δ_ρ and δ_κ and does not depend directly on ρ_1 and ρ_2 (or, κ_1 and κ_2) themselves (i.e., $CRB(\rho_i) = f(\delta_\rho, \delta_\kappa)$ and $CRB(\kappa_i) = f'(\delta_\rho, \delta_\kappa)$). Consequently, since the estimation accuracy depends only on the parameter separation, it is natural to expect that the SRL does not depend on ρ_c or κ_c . Indeed, and as expected, from Fig. 1 one notices that the SRLs using the exact values ρ_c and κ_c and the estimated values $\hat{\rho}_c$ and $\hat{\kappa}_c$ are the same. One concludes that the assumption A2 is not restrictive at all.
- On the other hand we consider now the ratio of SNR_o , given in (37), over the SNR, given in (36). Assuming the same signal sources power in the orthogonal and non-orthogonal cases, one obtains

$$\frac{SNR_o}{SNR} = \frac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2 - 2\Re\{\mathbf{s}_2^H \mathbf{s}_1\}}{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2}.$$

Consequently, in the context of orthogonal signal vectors, it should be noted that the minimum SNR in (36) may be either greater than or less than SNR_o in (37). For example, in the case of Binary Phase-Shift Keying (BPSK) $SNR_o > SNR$ as shown in Fig. 2. The gain is around 3 dB. The necessary and sufficient condition to have $SNR_o < SNR$ is $\Re\{\mathbf{s}_2^H \mathbf{s}_1\} > 0$.

- Finally, we study the impact of nonuniform array geometries on the SRL. Different configurations are considered herein as shown in Table 1; type 1 configuration where the three missing sensors cause a diminution of the array aperture; type 2 and type 3 two any configurations where the three missing sensors do not affect the array aperture; and the filled ULA configuration. From Fig. 3, one can deduce that a loss of sensors has an important impact on the SRL if the sensors are located in the extremity of the array (this loss is around 2.5 dB). However, this problem is largely mitigated if the missing sensors do not modify the

Table 1

Different array geometries where \bullet and \circ denote the position of sensors and missing sensors, respectively. The inter-element distance is $d = \nu/4$.

Array type	Array configuration								
Type 1	\circ	\bullet	\bullet	\circ	\bullet	\bullet	\bullet	\bullet	\circ
Type 2	\bullet	\circ	\bullet	\circ	\bullet	\circ	\bullet	\bullet	\bullet
Type 3	\bullet	\circ	\circ	\circ	\bullet	\bullet	\bullet	\bullet	\bullet
ULA	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet

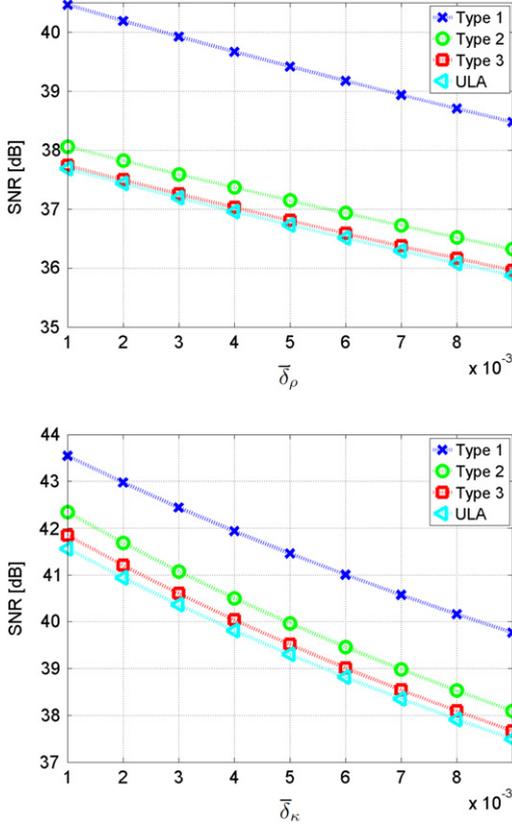


Fig. 3. The required SNR to resolve two known closely spaced near-field sources for different array geometries (see Table 1) with $L = 100$ snapshots, (top) for a fixed $\delta_\kappa = 0.001$, (bottom) for a fixed $\delta_\rho = 0.001$.

array aperture. Nevertheless, note that removal of sensors which are closer to the reference sensor (first sensor) causes a smaller reduction in f_i , (for $i = 2, 3, 4$) in (36), and hence a smaller increase in the required SNR.

5. Conclusion

In this paper, we have derived the Statistical Resolution Limit (SRL) for two closely spaced near-field time-varying narrowband known sources observed by a linear array (possibly nonuniform). Toward this goal, we have conducted a first-order Taylor expansion of the observation model and

a Generalized Likelihood Ratio Test (GLRT) based on a Constrained Maximum Likelihood Estimator (CMLE) of the SRL. This analysis provides useful information concerning the behavior of the SRL and the minimum SNR required to resolve two closely spaced near-field sources for a given probability of false alarm and a given probability of detection. In this way, the SRL has been analyzed with respect to the power signal sources distribution and the array aperture.

Appendix

The aim of this appendix is to find the distribution of $T(\mathbf{z})$ under \mathcal{H}_0 and \mathcal{H}_1 . Toward this end, we first begin by deriving the covariance matrix of $\Re\{\mathbf{D}^H \mathbf{z}\}$ denoted by $\mathbf{C}_{\Re\{\mathbf{D}^H \mathbf{z}\}}$. Since

$$\Re\{\mathbf{D}^H \mathbf{z}\} \sim \mathcal{N}(E\{\Re\{\mathbf{D}^H \mathbf{z}\}\}, \mathbf{C}_{\Re\{\mathbf{D}^H \mathbf{z}\}}),$$

one has

$$\mathbf{C}_{\Re\{\mathbf{D}^H \mathbf{z}\}} = E\{\Re\{\mathbf{D}^H \mathbf{v}\} \Re\{\mathbf{D}^H \mathbf{v}\}^T\} = E\{\boldsymbol{\alpha} \boldsymbol{\beta}^T\}, \tag{38}$$

where

$$\boldsymbol{\alpha} = \Re\{\mathbf{D}^H\} \Re\{\mathbf{v}\} - \Im\{\mathbf{D}^H\} \Im\{\mathbf{v}\} \tag{39}$$

and

$$\boldsymbol{\beta}^T = \Re\{\mathbf{v}^T\} \Re\{\mathbf{D}^*\} - \Im\{\mathbf{v}^T\} \Im\{\mathbf{D}^*\}. \tag{40}$$

Since \mathbf{v} is a complex white Gaussian circular noise, thus

$$E\{\Re\{\mathbf{v}\} \Re\{\mathbf{v}\}^T\} = E\{\Im\{\mathbf{v}\} \Im\{\mathbf{v}\}^T\} = \frac{\sigma^2}{2} \mathbf{I} \tag{41}$$

and

$$E\{\Re\{\mathbf{v}\} \Im\{\mathbf{v}\}^T\} = E\{\Im\{\mathbf{v}\} \Re\{\mathbf{v}\}^T\} = \mathbf{0}. \tag{42}$$

Thus, (38) becomes

$$\mathbf{C}_{\Re\{\mathbf{D}^H \mathbf{z}\}} = \frac{\sigma^2}{2} (\Re\{\mathbf{D}^H\} \Re\{\mathbf{D}^*\} + \Im\{\mathbf{D}^H\} \Im\{\mathbf{D}^*\}) = \frac{\sigma^2}{2} \Re\{\mathbf{D}^H \mathbf{D}\}. \tag{43}$$

Consequently, since $\Re\{\mathbf{D}^H \mathbf{D}\} = \mathbf{D}^H \mathbf{D}$ (see (21)), thus, $T(\mathbf{z})$ can be written as

$$T(\mathbf{z}) = \dot{\mathbf{z}}^T \mathbf{C}_{\dot{\mathbf{z}}}^{-1} \dot{\mathbf{z}}, \tag{44}$$

where the Gaussian random variable $\dot{\mathbf{z}}$ is given by $\dot{\mathbf{z}} = \Re\{\mathbf{D}^H \mathbf{z}\}$ and $\mathbf{C}_{\dot{\mathbf{z}}}$ denotes the covariance matrix of the random variable $\dot{\mathbf{z}}$. Thus, from (44), one can notice that

$$T(\mathbf{z}) \sim \chi_2^2(\lambda(P_{fa}, P_d)), \tag{45}$$

where $\chi_2^2(\lambda(P_{fa}, P_d))$ denotes the non-central distribution of two degrees of freedom in which the non-centrality parameter is given by

$$\lambda(P_{fa}, P_d) = E\{\dot{\mathbf{z}}\}^T \mathbf{C}_{\dot{\mathbf{z}}}^{-1} E\{\dot{\mathbf{z}}\} = \frac{2 \|\mathbf{s}_-\|^2}{\sigma^2} \bar{\boldsymbol{\delta}}^T \mathbf{F} \bar{\boldsymbol{\delta}}. \tag{46}$$

Finally, one obtains

$$T(\mathbf{z}) \sim \begin{cases} \chi_2^2(0) = \chi_2^2 & \text{under } \mathcal{H}_0, \\ \chi_2^2(\lambda(P_{fa}, P_d)) & \text{under } \mathcal{H}_1, \end{cases}$$

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