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# On the asymptotic resolvability of two point sources in known subspace interference using a GLRT-based framework

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## ABSTRACT

The asymptotic statistical resolution limit (SRL), denoted by  $\delta$ , characterizing the minimal separation to resolve two closely spaced far-field narrowband sources for a large number of observations, among a total number of  $M \geq 2$ , impinging on a linear array is derived. The two sources of interest (SOI) are corrupted by (1) the interference resulting from the  $M-2$  remaining sources and by (2) a broadband noise. Toward this end, a hypothesis test formulation is conducted. Depending on the *a priori* knowledge on the SOI, on the interfering sources and on the noise variance, the (constrained) maximum likelihood estimators (MLEs) of the SRL subject to  $\delta \in \mathbb{R}$  and/or in the context of the matched subspace detector theory are derived. Finally, we show that the SRL which is the minimum separation that allows a correct resolvability for given probabilities of false alarm and of detection can always be linked to a particular form of the Cramér–Rao bound (CRB), called the interference CRB (I-CRB), which takes into account the  $M-2$  interfering sources. As a by product, we give the theoretical expression of the minimum signal-to-interference-plus-noise ratio (SINR) required to resolve two closely spaced sources for several typical scenarios.

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## 1. Introduction

The context of narrowband far-field source localization has been widely investigated in the literature [1]. However, the ultimate performance in terms of resolution limit have not been fully investigated. The statistical resolution limit (SRL) [2–12], defined as the minimal separation between two signals in terms of parameter of interest, is a challenging problem [13] and an essential tool to quantify estimator performance. A closely related

problem is to derive the minimum signal-to-noise ratio (SNR) required to resolve two closely spaced sources.

Among all the different approaches to characterize the SRL, one can find three families. The first and oldest one is based on the null spectrum [2,3]. A second one is based on the estimation accuracy [5–8] and the last one and maybe the most promising one is based on detection theory in the context of the hypothesis test formulation. One can find in the literature several works related to the SRL or to the minimum SNR required to resolve two closely spaced sources using a hypothesis test formulation [9–12,14]. Specifically, in [11], Liu and Nehorai have derived the so-called statistical angular resolution limit (*i.e.*, the SRL) *w.r.t.* direction-of-arrival (DOA) using the asymptotic equivalence expression (in terms of number snapshots) of the Generalized Likelihood Ratio Test (GLRT). More recently, Amar and Weiss [12] have proposed to determine the SRL of two complex sinusoids with nearby frequencies using the

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Bayesian approach for a given correct decision probability. Finally, Shahram and Milanfar [10] have considered the resolvability of sinusoids with nearby frequencies by deriving the theoretical expressions of the minimum SNR required to resolve two closely spaced sources based on the GLRT.

In this paper, we focus our analysis on a GLR based hypothesis test formulation. This choice is motivated by the following arguments: (1) unlike the SRL based on the mean null spectrum [2,3], the SRL based on detection theory is claimed to be appropriate for all high-resolution algorithms since it is not related to a specific algorithm, (2) there exists a relationship between the SRL based on the estimation accuracy [7] (i.e., the Cramér–Rao bound (CRB)) and the SRL based on detection theory [11] (see Sections 3.4, 4.4, 5.4 and 6.4), (3) unlike the Bayesian approach, the use of the GLRT does not require any *prior* knowledge on the parameter of interest, (4) since the separation term is unknown to the user, it is impossible to design an optimal detector in the Neyman–Pearson sense [15] but the GLRT applied to our model is Uniformly Most Powerful invariant (UMP-invariant) test among all the invariant statistical tests [16], which is considered as the strongest statement of optimality that one could expect to obtain [17].

Note that all existing work in the field has been derived for the case of only two sources of interest (SOI) neglecting the effect of the other (interfering) sources. Another important point is that most of the contributions have been made in the case of spectral analysis and thus the impact of the array geometry on the resolution has not been studied in the context of statistical array processing with complex sources.

In this work, the considered model can be described by two narrowband far-field closely spaced SOI, among a total number of  $M \geq 2$  sources, embedded in a competitive environment constituted by (1) the interference resulting from the  $M-2$  remaining sources (called here sources of interference (SI)) and by (2) a broadband noise. For this general model, we derive the asymptotic (in term of the number of observations) theoretical expressions of the minimum signal-to-interference-plus-noise ratio (SINR) required to resolve two closely spaced sources for linear arrays.

The paper is organized as follows. We first begin by introducing the observation model and the problem setup in Section 2. Sections 3–6 are devoted to the derivation of the minimum SNR/SINR required to resolve two closely spaced sources and SRL derivations depending on the assumptions on the SOI, on the SI and on the noise variance. Section 7 gives a summary of the main results and compares the minimum SINRs required to resolve two closely spaced sources. Furthermore, in Section 8, numerical simulations are given to assess the effect of the array geometry, of the aperture, of the *prior* sources knowledge and the effect of the SI. Finally, Section 9 concludes this work.

## 2. Problem setup and assumptions

Let us consider a linear array (LA) with  $N$  sensors that receives at the  $t$ th snapshot, a signal emitted by  $M$  deterministic far-field and narrow-band sources, denoted by  $\{s_1(t), \dots, s_M(t)\}$ . For the  $n$ th sensor and for the  $t$ th

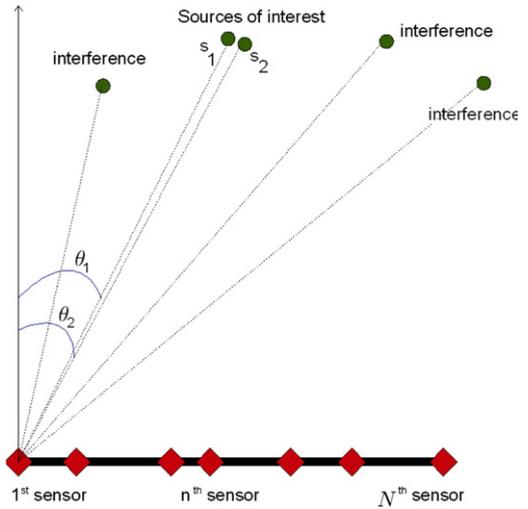


Fig. 1. Two closely spaced SOI imbedded in three interfering sources observed by a linear array with  $N=7$  sensors.

snapshot, the observation model is given by [1]

$$y_n(t) = \sum_{m=1}^M s_m(t) \exp(j\omega_m d_n) + v_n(t),$$

$$t = 1, \dots, L, n = 0, \dots, N-1, \quad (1)$$

where  $L$  stands for the number of snapshots,  $\omega_m = -2\pi \sin(\theta_m)/v$  is the parameter of interest of the  $m$ th source which is a function of the bearing  $\theta_m$  and of the signal wavelength  $v$ .  $d_n$  stands for the (known) distance between a reference sensor (the first sensor herein) and the  $n$ th sensor.<sup>1</sup> The additive noise  $v_n(t)$  is assumed to be a complex random process. Consequently, the observation vector at the  $t$ th snapshot, can be expressed as

$$\mathbf{y}(t) = [y_0(t) \dots y_{N-1}(t)]^T = [\mathbf{a}_1 \dots \mathbf{a}_M] \mathbf{s}(t) + \mathbf{v}(t),$$

where  $\mathbf{v}(t) = [v_0(t) \dots v_{N-1}(t)]^T$  and  $\mathbf{s}(t) = [s_1(t) \dots s_M(t)]^T$ , in which the  $(n+1)$ th entry of the steering vector  $\mathbf{a}_m$  is given by  $[\mathbf{a}_m]_{n+1} = \exp(j\omega_m d_n)$ ,  $m = 1, 2, \dots, M$ . Finally, the full observation vector is as follows:

$$\mathbf{y} \triangleq [\mathbf{y}^T(1) \mathbf{y}^T(2) \dots \mathbf{y}^T(L)]^T.$$

### 2.1. Hypothesis test formulation of the SRL in subspace interference

#### 2.1.1. SRL in subspace interference

The aim of this work is to derive the theoretical SINR required to resolve two SOI and the SRL, denoted by  $\delta$ , in the context of the scenario depicted in Fig. 1. More precisely,

1. Two closely spaced sources are of interest (SOI). Without loss of generality, we consider that these two sources are  $\mathbf{s}_1$  and  $\mathbf{s}_2$  (such that  $\mathbf{s}_1 \neq \mathbf{s}_2$ ). Consequently, the SRL (i.e., the separation) is defined as  $\delta \triangleq \omega_2 - \omega_1$ .
2. The  $M-2$  remaining sources, denoted by  $\{\mathbf{s}_3, \dots, \mathbf{s}_M\}$ ,

<sup>1</sup> For instance, in the case of the uniform LA (ULA),  $d_n = nd$  where  $d$  denotes the inter-element space between two successive sensors.

are viewed as an interference (called here sources of interference (SI)). The subspace interference of the SI is assumed known (*i.e.*, the DOAs of the SI are assumed known). However, we will consider the case of known and unknown signal SI  $\{\mathbf{s}_3, \dots, \mathbf{s}_M\}$ .

3. The background broadband noise is assumed to be a complex circular white Gaussian random process with zero-mean and variance  $\sigma^2$ .

Consequently, the problem setup can be viewed as deriving the theoretical SINR required to resolve two SOI and the SRL for two closely spaced SOI imbedded in a structured interference (the  $M-2$  remaining sources) and an unstructured interference (*i.e.*, the broadband noise).

### 2.1.2. Hypothesis test formulation

The problem of resolving *only* two closely spaced SOI in the context of a binary hypothesis test has been already (partially) tackled<sup>2</sup> in [9–12]. But in these references, the model is restricted to *only* two closely spaced SOI. In this work, we consider a more general model described in the previous section. This model has a rich structure since the impact of  $M-2$  interfering sources is taken into account. Consequently, the theoretical SINR required to resolve two SOI and the SRL derived in this paper are appropriate to the realistic situation where  $M \geq 2$  sources belong to the field of view of a LA (see Fig. 1).

In the following we assume that two sources are in the vicinity of each other. Let the hypothesis  $\mathcal{H}_0$  represents the case where the two SOI exist but are combined into a single signal, whereas the hypothesis  $\mathcal{H}_1$  embodies the situation where the two SOI are resolvable. Consequently, a convenient binary hypothesis test (see [9–12]) is given by

$$\begin{cases} \mathcal{H}_0 : \delta = 0, \\ \mathcal{H}_1 : \delta \neq 0. \end{cases} \quad (2)$$

Since the separation term  $\delta$  is unknown, it is impossible to design an optimal detector in the Neyman-Pearson sense. In this case, the Generalized Likelihood Ratio Test (GLRT) [15] is a well-known statistic test to solve such a problem and is given by

$$G(\mathbf{y}) = \frac{\max_{\delta, \boldsymbol{\rho}_1} p(\mathbf{y}; \delta, \boldsymbol{\rho}_1, \mathcal{H}_1)}{\max_{\boldsymbol{\rho}_0} p(\mathbf{y}; \boldsymbol{\rho}_0, \mathcal{H}_0)} = \frac{p(\mathbf{y}; \hat{\delta}, \hat{\boldsymbol{\rho}}_1, \mathcal{H}_1)}{p(\mathbf{y}; \hat{\boldsymbol{\rho}}_0, \mathcal{H}_0)} \stackrel{\mathcal{H}_1}{\geq} \eta', \quad (3)$$

in which  $p(\mathbf{y}; \boldsymbol{\rho}_0, \mathcal{H}_0)$  and  $p(\mathbf{y}; \delta, \boldsymbol{\rho}_1, \mathcal{H}_1)$  denote the probability density functions (pdf) under  $\mathcal{H}_0$  and  $\mathcal{H}_1$ , respectively, and where  $\eta'$ ,  $\hat{\delta}$  and  $\hat{\boldsymbol{\rho}}_i$  denote the detection threshold, the maximum likelihood estimate (MLE) of  $\delta$  under  $\mathcal{H}_1$  and the MLE of the parameter vector  $\boldsymbol{\rho}_i$  (containing all the unknown nuisance and/or unwanted parameters) under  $\mathcal{H}_i, i = 0, 1$ . If the statistic  $G(\mathbf{y})$  is greater than a given threshold  $\eta'$ , then the signals are said to be resolvable.

Unfortunately, closed-form expressions of  $\hat{\delta}$ ,  $\hat{\boldsymbol{\rho}}_1$  and  $\hat{\boldsymbol{\rho}}_0$  are not available (this is mainly due to the derivation of  $\hat{\delta}$  which is, in this case, a highly nonlinear and intractable optimization problem [18]). However, since the two SOI

are closely spaced (this assumption can be argued by the fact that the high resolution algorithms have asymptotically an infinite resolving power [19,9,7,10–12]), one can approximate model (1) into a new model which is linear *w.r.t.* the parameter  $\delta$ .

### 2.2. Linear form of the model with subspace interference

First, let us introduce the center parameter  $\omega_c = (\omega_1 + \omega_2)/2$ . As in [9–12], using the assumption of a small angular separation of the two SOI, a first-order Taylor expansion around  $\delta = 0$  leads to

$$\mathbf{y} \triangleq \mathbf{A}\mathbf{s}_+ + \delta\mathbf{B}\mathbf{s}_- + \mathbf{e} + \mathbf{v}, \quad (4)$$

where  $\mathbf{e} = \mathbf{C}\mathbf{s}$ ,  $\mathbf{C} = [\mathbf{A}_3 \dots \mathbf{A}_M]$ ,  $\mathbf{s} = [\mathbf{s}_3^T \dots \mathbf{s}_M^T]^T$  and

$$\mathbf{s}_+ = \mathbf{s}_1 + \mathbf{s}_2, \quad (5)$$

$$\mathbf{s}_- = \mathbf{s}_2 - \mathbf{s}_1, \quad (6)$$

in which  $\mathbf{s}_i = [s_i(1) \dots s_i(L)]^T$ . Furthermore, the signal  $\mathbf{e}$  encompasses the interference of all the sources apart from the two closest ones (*i.e.*, the first and the second one). In addition, denoting  $\mathbf{I}_L$  the  $L \times L$  identity matrix,  $\mathbf{d} = [d_0 \ d_1 \ \dots \ d_{N-1}]^T$  and  $\mathbf{a}$  the steering vector considered for the center parameter  $\omega_c$  (*i.e.*,  $[\mathbf{a}]_{n+1} \triangleq \exp(j\omega_c d_n)$ ,  $n = 0, \dots, N-1$ ), we define

$$\mathbf{A} \triangleq \begin{bmatrix} \mathbf{a} & \mathbf{0} \\ & \ddots \\ \mathbf{0} & \mathbf{a} \end{bmatrix}_{(NL) \times L} = \mathbf{I}_L \otimes \mathbf{a}, \quad (7)$$

$$\mathbf{B} \triangleq \frac{j}{2} \mathbf{I}_L \otimes \mathbf{a} \quad \text{where } \mathbf{a} \triangleq \mathbf{a} \odot \mathbf{d}, \quad (8)$$

$$\mathbf{A}_m \triangleq \mathbf{I}_L \otimes \mathbf{a}_m \quad \text{for } m = 3, \dots, M, \quad (9)$$

where  $\otimes$  and  $\odot$  stand for the Kronecker and the Hadamard products, respectively.

In the rest of the paper, and as in [10,11], the parameter  $\omega_c$  is assumed to be known or previously estimated. Furthermore, we consider that the matrix  $\mathbf{C}$  is known or previously estimated [20] (*i.e.*, the DOAs of the SI are known). Note that the case of unknown  $\omega_c$  and/or unknown  $\mathbf{C}$  leads to an untractable solution of the GLRT and, consequently, is beyond the scope of this paper.

### 2.3. Definition of the SNR and the SINR

A standard measure for the point source without interference is the signal-to-noise ratio defined as

$$\text{SNR} \triangleq \frac{\sum_{m=1}^2 \|\mathbf{s}_m\|^2}{\sigma^2}. \quad (10)$$

But a more convenient measure in case of interference is the signal to interference plus noise ratio defined according to

$$\text{SINR} \triangleq \frac{\sum_{m=1}^2 \|\mathbf{s}_m\|^2}{\|\mathbf{s}\|^2 + \sigma^2}. \quad (11)$$

Obviously, we have  $\text{SINR} \leq \text{SNR}$ . Let  $\text{INR} \triangleq \|\mathbf{s}\|^2 / \sigma^2$  be the interference to noise ratio. Straightforward relations

<sup>2</sup> More precisely, in this paper we consider the effect of the interference on the SRL and we derive analytical expressions of the minimum required SNR/SINR to resolve two closely spaced sources.

between the SNR and the SINR are

$$\text{SINR} = \frac{1}{1+\alpha} \text{SNR} \quad \text{if } \text{INR} = \alpha, \quad (12)$$

$$\text{SINR} = \text{SNR} \quad \text{if } \text{INR} = 0, \quad (13)$$

$$\text{SINR} \approx \text{SNR} \quad \text{if } \text{INR} \ll 1. \quad (14)$$

In this work, the theoretical SINR and SNR are derived for a given SRL in the scenarios listed above. More precisely, relation (12) means that if the INR is fixed to the constant  $\alpha$  then the SINR is proportional to the SNR. Relation (13) is a particular case of relation (12) and stands for the situation where there is no subspace interference. The last relation (14) means that the subspace interference is dominated by the noise.

### 3. Case 1: known SOI, known SI and known noise variance

First, let us consider the scenario where the two SOI, the SI and the noise variance are known, i.e.,  $\mathbf{s}_1$ ,  $\mathbf{s}_2$ ,  $\mathbf{s}$  and  $\sigma^2$  are known parameters. Let  $\delta_1$  be the SRL for the considered case. We define the new observation vector  $\mathbf{z} \triangleq \mathbf{y} - \mathbf{A}\mathbf{s}_+ - \mathbf{C}\mathbf{s}$ .

#### 3.1. Binary hypothesis test

With the aforementioned framework, the hypothesis test (2) becomes

$$\begin{cases} \mathcal{H}_0 : \mathbf{z} = \mathbf{v} \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I}), \\ \mathcal{H}_1 : \mathbf{z} = \mathbf{w}\delta_1 + \mathbf{v} \sim \mathcal{CN}(\mathbf{w}\delta_1, \sigma^2 \mathbf{I}), \end{cases} \quad (15)$$

in which  $\mathbf{w} \triangleq \mathbf{B}\mathbf{s}_-$ .

#### 3.2. Constrained MLE (CMLE) and GLRT

As  $\delta_1 \in \mathbb{R}$ , one has to find the constrained MLE (CMLE) of  $\delta_1$ . More precisely, the constrained optimization problem can be written according to

$$\arg \min_{\delta_1} L(\mathbf{z}, \delta_1) \quad \text{subject to } \delta_1 \in \mathbb{R},$$

where  $L(\mathbf{z}, \delta_1)$  is the negative log-likelihood function. According to Appendix A.1, the CMLE is

$$\hat{\delta}_1 = \frac{\Re\{\mathbf{w}^H \mathbf{z}\}}{\|\mathbf{w}\|^2}, \quad (16)$$

where  $\|\mathbf{w}\|^2 = \mathbf{s}_-^H \mathbf{B}^H \mathbf{B} \mathbf{s}_- = \frac{1}{4} \|\mathbf{s}_-\|^2 \|\hat{\mathbf{a}}\|^2$ . Using (16), the statistic of the GLRT is then given by

$$G(\mathbf{z}) = \frac{p(\mathbf{z}; \hat{\delta}_1, \mathcal{H}_1)}{p(\mathbf{z}; \mathcal{H}_0)} = \exp\left(\frac{1}{\sigma^2} (\|\mathbf{z}\|^2 - \|\mathbf{z} - \mathbf{w}\hat{\delta}_1\|^2)\right) \stackrel{\mathcal{H}_1}{\gtrsim} \eta'. \quad (17)$$

Plugging (16) into (17), and defining a new statistic, denoted by  $T(\mathbf{z})$ , one obtains

$$\begin{aligned} T(\mathbf{z}) &\triangleq 2 \ln G(\mathbf{z}) = \frac{2}{\sigma^2} (\hat{\delta}_1 \mathbf{z}^H \mathbf{w} + \hat{\delta}_1 \mathbf{w}^H \mathbf{z} - \hat{\delta}_1^2 \|\mathbf{w}\|^2) \\ &= \frac{2\Re\{\mathbf{w}^H \mathbf{z}\}}{\sigma^2 \|\mathbf{w}\|^2} (\mathbf{w}^H \mathbf{z} + \mathbf{z}^H \mathbf{w} - \Re\{\mathbf{w}^H \mathbf{z}\}) = \frac{\Re^2\{\mathbf{w}^H \mathbf{z}\}}{\|\mathbf{w}\|^2 \frac{\sigma^2}{2}}. \end{aligned} \quad (18)$$

Using the result of Appendix B, we have

$$T(\mathbf{z}) \sim \begin{cases} \chi_1^2 & \text{under } \mathcal{H}_0, \\ \chi_1^2(\lambda_1(P_{fa}, P_d)) & \text{under } \mathcal{H}_1, \end{cases}$$

where the non-centrality parameter is given by

$$\lambda_1(P_{fa}, P_d) = \frac{2\delta_1^2 \|\mathbf{w}\|^2}{\sigma^2} = \frac{\delta_1^2}{2\sigma^2} \|\mathbf{s}_-\|^2 \|\hat{\mathbf{a}}\|^2, \quad (19)$$

and where  $\chi_1^2$  denotes the central distribution with one degree of freedom. Since  $G(\mathbf{z}) \stackrel{\mathcal{H}_1}{\gtrsim} \eta' \Leftrightarrow T(\mathbf{z}) \stackrel{\mathcal{H}_1}{\gtrsim} \eta \triangleq \ln \eta'$ , the probability of false alarm and the probability of detection are then given by  $P_{fa} = Q_{\chi_1^2}(\eta)$  and  $P_d = Q_{\chi_1^2(\lambda_1(P_{fa}, P_d))}(\eta)$ , respectively, in which  $Q_{\chi_1^2}(\cdot)$  and  $Q_{\chi_1^2(\lambda_1(P_{fa}, P_d))}(\cdot)$  denote the right tail of the  $\chi_1^2$  pdf and the  $\chi_1^2(\lambda_1(P_{fa}, P_d))$  pdf, respectively. In practice  $\lambda_1(P_{fa}, P_d)$  can be derived for a given  $P_{fa}$  and  $P_d$  as the solution of  $Q_{\chi_1^2}^{-1}(P_{fa}) = Q_{\chi_1^2(\lambda_1(P_{fa}, P_d))}^{-1}(P_d)$ .

#### 3.3. Theoretical SINR derivation

**Result 1.** The minimum SINR required to resolve two closely spaced known SOI (w.r.t. the SRL  $\delta_1$ ) imbedded in  $M-2$  known sources in a known noise variance, is given by

$$\text{SINR}_1 = \frac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2}{\|\mathbf{s}\|^2 + \delta_1^2 \frac{2\|\mathbf{w}\|^2}{\lambda_1(P_{fa}, P_d)}}. \quad \square \quad (20)$$

**Result 2.** The minimum SNR (w.r.t. the SRL  $\delta_1$ ) required to resolve two closely spaced known SOI in a known noise variance is given by

$$\text{SNR}_1 = \lambda_1(P_{fa}, P_d) \frac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2}{2\delta_1^2 \|\mathbf{w}\|^2}. \quad \square \quad (21)$$

**Result 3.** If the SOI are orthogonal signals, i.e.,  $\mathbf{s}_1^H \mathbf{s}_2 = \mathbf{s}_2^H \mathbf{s}_1 = 0$ , then  $\|\mathbf{s}_-\|^2 = \|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2$ . The minimum SNR required to resolve two closely spaced SOI is given by

$$\text{SNR}_{1o} = \frac{2\lambda_1(P_{fa}, P_d)}{\delta_1^2 \|\hat{\mathbf{a}}\|^2}. \quad \square \quad (22)$$

The last result means that the minimum SNR required to resolve two closely spaced SOI for orthogonal SOI is invariant to the source powers.

#### 3.4. Alternative expression of the non-centrality parameter

Let  $[\delta_1 \sigma^2]^T$  be the vector collecting the parameters under  $\mathcal{H}_1$ . The associated CRB( $\delta_1$ ) which is derived in Appendix C.1 is given by

$$\text{CRB}(\delta_1) = \frac{\sigma^2}{2\|\mathbf{w}\|^2} = \frac{2\sigma^2}{\|\mathbf{s}_-\|^2 \|\hat{\mathbf{a}}\|^2}.$$

Consequently, (19) can be rewritten w.r.t. the CRB as

$$\lambda_1(P_{fa}, P_d) = \delta_1^2 \text{CRB}^{-1}(\delta_1). \quad (23)$$

Note that (23) is an expression where  $\delta_1$  is the unknown variable. This is in agreement with the formulation introduced in Ref. [7].

#### 4. Case 2: known SOI, unknown SI, known noise variance

##### 4.1. Binary hypothesis test

In the following, we consider the case where two known SOI are imbedded in  $M-2$  unknown interfering sources. In addition, the noise variance is assumed to be known. Consequently, the observations under each hypothesis are given by

$$\begin{cases} \mathcal{H}_0 : \mathbf{z} \triangleq \mathbf{y} - \mathbf{A}\mathbf{s}_+ = \mathbf{C}\mathbf{s} + \mathbf{v} \sim \mathcal{CN}(\mathbf{C}\mathbf{s}, \sigma^2 \mathbf{I}), \\ \mathcal{H}_1 : \mathbf{z} = \delta_2 \mathbf{w} + \mathbf{C}\mathbf{s} + \mathbf{v} \sim \mathcal{CN}(\delta_2 \mathbf{w} + \mathbf{C}\mathbf{s}, \sigma^2 \mathbf{I}), \end{cases} \quad (24)$$

where  $\delta_2$  denotes the SRL.

##### 4.2. Joint CMLE and GLRT

As  $\delta_2 \in \mathbb{R}$ , one has to find jointly the CMLE of the SRL and the MLE of the interfering sources. Let us reorganize the observation according to  $\mathbf{z} = \mathbf{Q}\mathbf{p} + \mathbf{v}$  where  $\mathbf{Q} = [\mathbf{w} \ \mathbf{C}]$  and  $\mathbf{p} = [\delta_2 \ \mathbf{s}^T]^T$ , then the constrained optimization problem can be written according to

$$\arg \min_{\mathbf{p}} L(\mathbf{z}, \mathbf{p}) \quad \text{subject to } \mathbf{e}_1^T \mathbf{p} \in \mathbb{R}, \quad (25)$$

where  $\mathbf{e}_1 = [1 \ 0 \ \dots \ 0]^T$  and  $L(\mathbf{z}, \mathbf{p})$  is the negative log-likelihood function of the observation. According to Appendix A.3, we have

$$\hat{\delta}_2 = \frac{\Re\{\mathbf{w}^H \mathbf{P}_C^\perp \mathbf{z}\}}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2}, \quad (26)$$

$$\hat{\mathbf{s}}_{\mathcal{H}_0} = \mathbf{C}^\dagger \mathbf{z}, \quad (27)$$

$$\hat{\mathbf{s}}_{\mathcal{H}_1} = \mathbf{C}^\dagger (\mathbf{z} - \hat{\delta}_2 \mathbf{w}), \quad (28)$$

where  $\dagger$  stands for the Moore–Penrose pseudo-inverse [21]. Consequently using (26)–(28), one obtains

$$\begin{cases} \hat{\mathbf{v}}_{\mathcal{H}_0} = \mathbf{z} - \mathbf{C}\hat{\mathbf{s}}_{\mathcal{H}_0} = \mathbf{P}_C^\perp \mathbf{z} & \text{under } \mathcal{H}_0, \\ \hat{\mathbf{v}}_{\mathcal{H}_1} = \mathbf{z} - \hat{\delta}_2 \mathbf{w} - \mathbf{C}\hat{\mathbf{s}}_{\mathcal{H}_1} = \mathbf{P}_C^\perp \mathbf{z} - \mathbf{P}_C^\perp \mathbf{w} \frac{\Re\{\mathbf{w}^H \mathbf{P}_C^\perp \mathbf{z}\}}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2} & \text{under } \mathcal{H}_1. \end{cases} \quad (29)$$

Now, we are ready to use the statistic  $\tilde{T}(\mathbf{y})$  based on the GLRT which is defined as follows:

$$\tilde{T}(\mathbf{z}) \triangleq 2 \ln G(\mathbf{z}) = \frac{2}{\sigma^2} (\|\hat{\mathbf{v}}_{\mathcal{H}_0}\|^2 - \|\hat{\mathbf{v}}_{\mathcal{H}_1}\|^2). \quad (30)$$

Plugging (29) in (30) and after some calculus, one obtains

$$\tilde{T}(\mathbf{z}) = \frac{2}{\sigma^2} \frac{\Re^2\{\mathbf{w}^H \mathbf{P}_C^\perp \mathbf{z}\}}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2} = \frac{\Re^2\{\tilde{\mathbf{w}}^H \tilde{\mathbf{z}}\}}{\frac{\sigma^2}{2} \|\tilde{\mathbf{w}}\|^2}, \quad (31)$$

where  $\tilde{\mathbf{z}} = \tilde{\mathbf{U}}^H \mathbf{z}$  and where  $\tilde{\mathbf{w}} = \tilde{\mathbf{U}}^H \mathbf{w}$  in which  $\mathbf{P}_C^\perp = \tilde{\mathbf{U}} \tilde{\mathbf{U}}^H$  is any orthogonal decomposition for which  $\tilde{\mathbf{U}}^H \tilde{\mathbf{U}} = \mathbf{I}$  [22, Eq (A.4.7)]. The statistics of the random variable  $\tilde{\mathbf{z}}$  follow:

$$\begin{cases} \tilde{\mathbf{z}}_{\mathcal{H}_0} \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I}) & \text{under } \mathcal{H}_0, \\ \tilde{\mathbf{z}}_{\mathcal{H}_1} \sim \mathcal{CN}(\tilde{\mathbf{w}} \delta_2, \sigma^2 \mathbf{I}) & \text{under } \mathcal{H}_1, \end{cases} \quad (32)$$

since  $\tilde{\mathbf{U}}^H \mathbf{C} = \mathbf{0}$ . Expressions (31) and (32) are formally similar to the case studied in Appendix B. So, we can

conclude that

$$\tilde{T}(\mathbf{z}) \sim \begin{cases} \chi_1^2 & \text{under } \mathcal{H}_0, \\ \chi_1^2(\lambda_2(P_{fa}, P_d)) & \text{under } \mathcal{H}_1, \end{cases} \quad (33)$$

where

$$\lambda_2(P_{fa}, P_d) = \frac{2\delta_2^2 \|\mathbf{P}_C^\perp \mathbf{w}\|^2}{\sigma^2}. \quad (34)$$

##### 4.3. Theoretical SINR derivation

From (34), one can state the following results:

**Result 4.** The minimum SINR (*w.r.t.* the SRL  $\delta_2$ ) required to resolve two closely spaced known SOI imbedded in  $M-2$  unknown sources in a known noise variance is given by

$$\text{SINR}_2 = \frac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2}{\|\mathbf{s}\|^2 + \delta_2^2 \frac{2\|\mathbf{P}_C^\perp \mathbf{w}\|^2}{\lambda_2(P_{fa}, P_d)}}. \quad \square \quad (35)$$

**Result 5.** The minimum SNR (*w.r.t.* the SRL  $\delta_2$ ) required to resolve two closely spaced known SOI with a known noise variance is obtained for  $\mathbf{P}_C^\perp = \mathbf{I}$  and  $\mathbf{s} = \mathbf{0}$  and thus is given by

$$\text{SNR}_2 = \lambda_2(P_{fa}, P_d) \frac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2}{2\delta_2^2 \|\mathbf{w}\|^2}. \quad \square \quad (36)$$

**Result 6.** If the SOI are orthogonal. The minimum SNR required to resolve two closely spaced SOI is given by

$$\text{SNR}_{20} = \frac{2\lambda_2(P_{fa}, P_d)}{\delta_2^2 \|\mathbf{a}\|^2}. \quad \square \quad (37)$$

The last result means that the minimum SNR required to resolve two closely spaced orthogonal SOI is invariant to the source powers.

##### 4.4. Alternative expression of the non-centrality parameter

In [23], the interference Cramér–Rao bound (I-CRB) has been introduced. This bound is the CRB for the unknown parameter  $[\delta_2 \ \sigma^2]^T$  and for the projected observation  $\tilde{\mathbf{z}} = \tilde{\mathbf{U}}^H \mathbf{z}$ . So, this bound integrates the subspace interference related to  $\langle \mathbf{C} \rangle$ . More precisely, this bound takes the following form:

$$\text{I-CRB}(\delta_2) = \frac{\sigma^2}{2\|\mathbf{P}_C^\perp \mathbf{w}\|^2}.$$

See Appendix C.2 for the proof. Consequently, using (34), one deduces the relationship between the I-CRB and the non-centrality parameter

$$\lambda_2(P_{fa}, P_d) = \delta_2^2 \text{I-CRB}^{-1}(\delta_2). \quad (38)$$

#### 5. Case 3: unknown SOI, unknown SI and known noise variance

##### 5.1. Binary hypothesis test

In the following, we consider the case where two unknown sources are imbedded in  $M-2$  unknown sources. The noise variance is assumed to be known. Consequently,

the observations under each hypothesis are given by

$$\begin{cases} \mathcal{H}_0 : \mathbf{y} = \mathbf{D}\mathbf{g} + \mathbf{v} \sim \mathcal{CN}(\mathbf{D}\mathbf{g}, \sigma^2\mathbf{I}), \\ \mathcal{H}_1 : \mathbf{y} = \mathbf{B}\boldsymbol{\theta} + \mathbf{D}\mathbf{g} + \mathbf{v} \sim \mathcal{CN}(\mathbf{B}\boldsymbol{\theta} + \mathbf{D}\mathbf{g}, \sigma^2\mathbf{I}), \end{cases} \quad (39)$$

where  $\mathbf{D} \triangleq [\mathbf{A} \ \mathbf{C}] \in \mathbb{C}^{NL \times (M-1)L}$ , in which the unknown vector parameters  $\boldsymbol{\theta}$  and  $\mathbf{g}$  are defined by

$$\boldsymbol{\theta} \triangleq \delta_3 \mathbf{s}_-,$$

$$\mathbf{g} \triangleq \begin{bmatrix} \mathbf{s}_+ \\ \mathbf{s} \end{bmatrix},$$

where  $\delta_3$  is the SRL.

## 5.2. Unconstrained MLE and GLRT

The unconstrained MLEs of the unknown parameters are given by [24]

$$\hat{\boldsymbol{\theta}} = (\mathbf{B}^H \mathbf{P}_D^\perp \mathbf{B})^{-1} \mathbf{B}^H \mathbf{P}_D^\perp \mathbf{y}, \quad (40)$$

$$\hat{\mathbf{g}}_{\mathcal{H}_0} = (\mathbf{D}^H \mathbf{D})^{-1} \mathbf{D}^H \mathbf{y}, \quad (41)$$

$$\hat{\mathbf{g}}_{\mathcal{H}_1} = (\mathbf{D}^H \mathbf{P}_B^\perp \mathbf{D})^{-1} \mathbf{D}^H \mathbf{P}_B^\perp \mathbf{y}, \quad (42)$$

where  $\mathbf{P}_D^\perp \triangleq \mathbf{I} - \mathbf{P}_D$ , in which  $\mathbf{P}_D$  denotes the orthogonal projector onto the subspace spanned by the columns of the matrix  $\mathbf{D}$ .

Consequently, the MLEs of the noise are

$$\begin{cases} \hat{\mathbf{v}}_{\mathcal{H}_0} = \mathbf{z} - \mathbf{D}\hat{\mathbf{g}} = \mathbf{P}_D^\perp \mathbf{y} & \text{under } \mathcal{H}_0, \\ \hat{\mathbf{v}}_{\mathcal{H}_1} = \mathbf{z} - \mathbf{B}\hat{\boldsymbol{\theta}} - \mathbf{D}\hat{\mathbf{g}} = (\mathbf{I} - \mathbf{E}_{BD} - \mathbf{E}_{DB})\mathbf{z} = \mathbf{P}_{\{\mathbf{B}, \mathbf{D}\}}^\perp \mathbf{y} & \text{under } \mathcal{H}_1, \end{cases} \quad (43)$$

where the oblique projectors  $\mathbf{E}_{BD}$  and  $\mathbf{E}_{DB}$  are defined as

$$\mathbf{E}_{BD} = \mathbf{B}(\mathbf{B}^H \mathbf{P}_D^\perp \mathbf{B})^{-1} \mathbf{B}^H \mathbf{P}_D^\perp, \quad (44)$$

$$\mathbf{E}_{DB} = \mathbf{D}(\mathbf{D}^H \mathbf{P}_B^\perp \mathbf{D})^{-1} \mathbf{D}^H \mathbf{P}_B^\perp. \quad (45)$$

Now, we are ready to use the statistic  $T'(\mathbf{y})$  based on the GLRT and defined as follows:

$$T'(\mathbf{y}) \triangleq 2 \ln G(\mathbf{y}) = \frac{2}{\sigma^2} (\|\hat{\mathbf{v}}_{\mathcal{H}_0}\|^2 - \|\hat{\mathbf{v}}_{\mathcal{H}_1}\|^2) \quad (46)$$

$$T'(\mathbf{y}) = \frac{1}{\sigma^2} \mathbf{y}^H (\mathbf{P}_D^\perp - \mathbf{P}_{\{\mathbf{B}, \mathbf{D}\}}^\perp) \mathbf{y}. \quad (47)$$

Using [16, Eq. (3.7)] and [24, Eq. (19)], one has  $\mathbf{P}_D^\perp - \mathbf{P}_{\{\mathbf{B}, \mathbf{D}\}}^\perp = \mathbf{P}_D^\perp \mathbf{E}_{BD} \mathbf{P}_D^\perp = \mathbf{P}_{\mathbf{P}_D^\perp \mathbf{B}}^\perp$ . Thus,

$$T'(\mathbf{y}) = \frac{2}{\sigma^2} \mathbf{y}^H \mathbf{P}_{\mathbf{P}_D^\perp \mathbf{B}} \mathbf{y}. \quad (48)$$

Let  $\mathbf{P}_{\mathbf{P}_D^\perp \mathbf{B}} = \mathbf{U}\mathbf{U}^H$  be any orthogonal decomposition [22] of the projector  $\mathbf{P}_{\mathbf{P}_D^\perp \mathbf{B}}$  such that  $\mathbf{U}^H \mathbf{U} = \mathbf{I}$  and define an auxiliary random variable  $\hat{\mathbf{y}} = \mathbf{U}^H \mathbf{y}$ . One should note that

$$\begin{cases} \hat{\mathbf{y}} = \mathbf{U}^H \mathbf{v} \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I}) & \text{under } \mathcal{H}_0, \\ \hat{\mathbf{y}} = \mathbf{U}^H \mathbf{B}\boldsymbol{\theta} + \mathbf{U}^H \mathbf{v} \sim \mathcal{CN}(\mathbf{U}^H \mathbf{B}\boldsymbol{\theta}, \sigma^2 \mathbf{I}) & \text{under } \mathcal{H}_1. \end{cases} \quad (49)$$

Consequently,

$$T'(\mathbf{y}) \sim \begin{cases} \chi_{2r}^2 & \text{under } \mathcal{H}_0, \\ \chi_{2r}^2(\lambda_3(P_{fa}, P_d)) & \text{under } \mathcal{H}_1, \end{cases} \quad (50)$$

where [17]  $r = \text{tr}(\mathbf{P}_{\mathbf{P}_D^\perp \mathbf{B}}) = \text{rank}(\mathbf{P}_{\mathbf{P}_D^\perp \mathbf{B}}) = L$  and

$$\begin{aligned} \lambda_3(P_{fa}, P_d) &= \frac{\boldsymbol{\theta}^H \mathbf{B}^H \mathbf{U}\mathbf{U}^H \mathbf{B}\boldsymbol{\theta}}{\sigma^2/2} = \frac{2\delta_3^2}{\sigma^2} \mathbf{s}_-^H \mathbf{B}^H \mathbf{P}_{\mathbf{P}_D^\perp \mathbf{B}} \mathbf{B} \mathbf{s}_- \\ &= \frac{2\delta_3^2}{\sigma^2} \|\mathbf{P}_D^\perp \mathbf{w}\|^2. \end{aligned} \quad (51)$$

Note that the non-centrality parameter  $\lambda_3(P_{fa}, P_d)$  can be numerically computed as the solution of  $\mathcal{Q}_{\chi_{2r}^2}^{-1}(P_{fa}) = \mathcal{Q}_{\chi_{2r}^2(\lambda_3(P_{fa}, P_d))}^{-1}(P_d)$ .

## 5.3. Theoretical SINR derivation

From (51), one can state the following results:

**Result 7.** The minimum SINR (w.r.t. the SRL  $\delta_3$ ) required to resolve two closely spaced unknown SOI imbedded in  $M-2$  unknown sources with a known noise variance is given by

$$\text{SINR}_3 = \frac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2}{\|\mathbf{s}\|^2 + \delta_3^2 \frac{2\|\mathbf{P}_D^\perp \mathbf{w}\|^2}{\lambda_3(P_{fa}, P_d)}}. \quad \square \quad (52)$$

**Result 8.** The minimum SNR (w.r.t. the SRL  $\delta_3$ ) required to resolve two closely spaced unknown SOI with a known noise variance is given by

$$\text{SNR}_3 = \lambda_3(P_{fa}, P_d) \frac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2}{2\delta_3^2 \|\mathbf{P}_A^\perp \mathbf{w}\|^2}. \quad \square \quad (53)$$

A straightforward derivation leads to

$$\|\mathbf{P}_A^\perp \mathbf{w}\|^2 = \frac{1}{4} \|\mathbf{s}_-\|^2 \|\mathbf{b}\|^2, \quad (54)$$

where  $\mathbf{b} = \hat{\mathbf{a}} - (\mathbf{a}^H \hat{\mathbf{a}}/L)\mathbf{a}$  and  $\|\mathbf{b}\|^2 = \|\hat{\mathbf{a}}\|^2 - |\mathbf{a}^H \hat{\mathbf{a}}|^2/L$ .

**Result 9.** If the SOI are orthogonal. The minimum SNR required to resolve two closely spaced SOI is given by

$$\text{SNR}_{3o} = \frac{2\lambda_3(P_{fa}, P_d)}{\delta_3^2 \|\mathbf{b}\|^2}. \quad \square \quad (55)$$

The last result means that the minimum SNR required to resolve two closely spaced orthogonal SOI is invariant to the source powers.

## 5.4. Alternative expression of the non-centrality parameter

In the case of unknown SOI, the FIM is not invertible, therefore the CRB of  $\delta_3$  does not exist. This arises due to the lack of identifiability in model (39) because of multiplicative ambiguity in the product  $\delta_3 \mathbf{s}_-$ . To obtain an invertible FIM, it is necessary to assume known SOI as in cases 1 and 2. Note that if, as in case 3 (and also the following case, i.e., case 4), the vector of interest is  $\boldsymbol{\theta} = \delta_3 \mathbf{s}_-$ , there is no ambiguity and it exists an unbiased estimator (and thus the CRB) of  $\boldsymbol{\theta}$ . Keeping in mind this fact, it is interesting to note that the I-CRB (as derived in Appendix C.2) for the unknown parameters  $\delta_3$  w.r.t. the interference subspace  $\langle \mathbf{D} \rangle$  and for known SOI is given by

$$\text{I-CRB}(\delta_3) = \frac{\sigma^2}{2\|\mathbf{P}_D^\perp \mathbf{w}\|^2}.$$

This can be linked to the non-centrality parameter in (51) according to

$$\lambda_3(P_{fa}, P_d) = \delta_3^2 \text{I-CRB}^{-1}(\delta_3). \quad (56)$$

## 6. Case 4: unknown SOI, unknown SI and unknown noise variance

### 6.1. Binary hypothesis test

In the following, we consider the general case where two unknown sources are imbedded in  $M-2$  unknown sources. In addition,  $\sigma^2$  is assumed to be unknown. Let  $\delta_4$  be the SRL. The observations under each hypothesis are given by

$$\begin{cases} \mathcal{H}_0 : \mathbf{y} = \mathbf{D}\mathbf{g} + \mathbf{v} \sim \mathcal{CN}(\mathbf{D}\mathbf{g}, \sigma^2 \mathbf{I}), \\ \mathcal{H}_1 : \mathbf{y} = \mathbf{B}\boldsymbol{\theta} + \mathbf{D}\mathbf{g} + \mathbf{v} \sim \mathcal{CN}(\mathbf{B}\boldsymbol{\theta} + \mathbf{D}\mathbf{g}, \sigma^2 \mathbf{I}). \end{cases} \quad (57)$$

### 6.2. The GLRT derivation

From (57), the GLRT is given by

$$G(\mathbf{y}) = \frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} = \frac{\|\hat{\mathbf{v}}_{\mathcal{H}_0}\|^2}{\|\hat{\mathbf{v}}_{\mathcal{H}_1}\|^2}, \quad (58)$$

where the MLE of the noise variance under each hypothesis is given by [25]

$$\hat{\sigma}_i^2 = \frac{1}{NL} \|\hat{\mathbf{v}}_{\mathcal{H}_i}\|^2. \quad (59)$$

After some straightforward derivations, we obtain

$$\begin{cases} \hat{\mathbf{v}}_{\mathcal{H}_0} = \mathbf{y} - \mathbf{D}\hat{\mathbf{g}}_{\mathcal{H}_0} = \mathbf{P}_{\mathbf{D}}^{\perp} \mathbf{y} & \text{under } \mathcal{H}_0, \\ \hat{\mathbf{v}}_{\mathcal{H}_1} = \mathbf{y} - \mathbf{B}\hat{\boldsymbol{\theta}}_{\mathcal{H}_1} - \mathbf{D}\hat{\mathbf{g}}_{\mathcal{H}_1} = (\mathbf{I} - \mathbf{E}_{\mathbf{B}\mathbf{D}} - \mathbf{E}_{\mathbf{D}\mathbf{B}}) \mathbf{y} = \mathbf{P}_{\mathbf{B}\mathbf{D}}^{\perp} \mathbf{y} & \text{under } \mathcal{H}_1, \end{cases} \quad (60)$$

and where  $\hat{\boldsymbol{\theta}}$ ,  $\hat{\mathbf{g}}_{\mathcal{H}_0}$  and  $\hat{\mathbf{g}}_{\mathcal{H}_1}$  are given by (40)–(42), respectively. In this case it is more convenient to define the statistic  $T''(\mathbf{y})$  as follows:

$$T''(\mathbf{y}) \triangleq (\ln G(\mathbf{y}))^{1/NL} - 1 = \frac{T'(\mathbf{y})}{N(\mathbf{y})}, \quad (61)$$

where  $N(\mathbf{y}) = (1/\sigma^2) \mathbf{y}^H \mathbf{P}_{\mathbf{B}\mathbf{D}}^{\perp} \mathbf{y}$ . In addition, using any orthogonal decomposition [22], one has  $\mathbf{P}_{\mathbf{B}\mathbf{D}}^{\perp} = \mathbf{U} \mathbf{U}^H$ . Consequently,  $N(\mathbf{y}) = \|\bar{\mathbf{y}}\|^2$ , in which  $\bar{\mathbf{y}} = \mathbf{U}^H \mathbf{y}$ . Thus,

$$T''(\mathbf{y}) = \frac{\|\hat{\mathbf{y}}\|^2}{\|\bar{\mathbf{y}}\|^2}, \quad (62)$$

and

$$\begin{cases} \bar{\mathbf{y}} = \mathbf{U}^H \mathbf{v} \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I}) & \text{under } \mathcal{H}_0, \\ \bar{\mathbf{y}} = \mathbf{U}^H \mathbf{v} \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I}) & \text{under } \mathcal{H}_1, \end{cases} \Rightarrow \begin{cases} N(\mathbf{y}) \sim \chi_{2r}^2 & \text{under } \mathcal{H}_0, \\ N(\mathbf{y}) \sim \chi_{2r}^2(0) & \text{under } \mathcal{H}_1, \end{cases}$$

where  $r' = \text{tr}(\mathbf{P}_{\mathbf{B}\mathbf{D}}^{\perp}) = NL - \text{rank}(\mathbf{P}_{\mathbf{B}\mathbf{D}}) = (N-M)L$ .

Furthermore, one can notice that the random variables  $\|\bar{\mathbf{y}}\|^2$  and  $\|\hat{\mathbf{y}}\|^2$  are independent (see Appendix D). Consequently, a new statistic  $V(\mathbf{y})$  is described as follows:

$$V(\mathbf{y}) \triangleq (N-M) T''(\mathbf{y}) \sim \begin{cases} F_{2L, 2(N-M)L} & \text{under } \mathcal{H}_0, \\ F_{2L, 2(N-M)L}(\lambda_4(P_{fa}, P_d)) & \text{under } \mathcal{H}_1, \end{cases} \quad (63)$$

where  $F_{2L, 2(N-M)L}$  and  $F_{2L, 2(N-M)L}(\lambda_4(P_{fa}, P_d))$  denote the  $F$  central and non-central distributions [15], respectively, of  $2L$  and  $2(N-M)L$  degrees of freedom, in which the non-centrality parameter is given by

$$\lambda_4(P_{fa}, P_d) = \frac{2\delta_4^2}{\sigma^2} \|\mathbf{P}_{\mathbf{D}}^{\perp} \mathbf{w}\|^2. \quad (64)$$

Once again, note that the non-centrality parameter  $\lambda_4(P_{fa}, P_d)$  can be computed numerically as the solution of  $\mathcal{Q}_{F_{2L, 2(N-M)L}}^{-1}(P_{fa}) = \mathcal{Q}_{F_{2L, 2(N-M)L}(\lambda_4(P_{fa}, P_d))}^{-1}(P_d)$  with  $2L$  and  $2(N-M)L$  degree of freedom, where  $\mathcal{Q}_{F_{2L, 2(N-M)L}}^{-1}(\beta)$  and  $\mathcal{Q}_{F_{2L, 2(N-M)L}(\lambda_4(P_{fa}, P_d))}^{-1}(\beta)$  denote the right tail of the pdf  $F_{2L, 2(N-M)L}$  and  $F_{2L, 2(N-M)L}(\lambda_4(P_{fa}, P_d))$ , respectively, starting at  $\beta$ .

### 6.3. Theoretical SINR derivation

From (64), one can state the following results:

**Result 10.** The minimum SINR (*w.r.t.* the SRL  $\delta_4$ ) required to resolve two closely spaced unknown SOI imbedded in  $M-2$  unknown sources with an unknown noise variance is given by

$$\text{SINR}_4 = \frac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2}{\|\mathbf{s}\|^2 + \delta_4^2 \frac{2\|\mathbf{P}_{\mathbf{D}}^{\perp} \mathbf{w}\|^2}{\lambda_4(P_{fa}, P_d)}}. \quad \square \quad (65)$$

**Result 11.** The minimum SNR required (*w.r.t.* the SRL  $\delta_4$ ) required to resolve two closely spaced unknown SOI with an unknown noise variance is given by

$$\text{SNR}_4 = \lambda_4(P_{fa}, P_d) \frac{\|\mathbf{s}_1\|^2 + \|\mathbf{s}_2\|^2}{2\delta_4^2 \|\mathbf{P}_{\mathbf{A}}^{\perp} \mathbf{w}\|^2}. \quad \square \quad (66)$$

As in case 3, the minimum SNR required to resolve two closely spaced orthogonal SOI is given by the following result:

**Result 12.** If the SOI are orthogonal. The minimum SNR required to resolve two closely spaced SOI is given by

$$\text{SNR}_{4o} = \frac{2\lambda_4(P_{fa}, P_d)}{\delta_4^2 \|\mathbf{b}\|^2}. \quad \square \quad (67)$$

The last result means that the minimum SNR required to resolve two closely spaced orthogonal SOI is invariant to the source powers.

### 6.4. Alternative expression of the non-centrality parameter

As in case 3, the I-CRB defined in Appendix C.2 for the unknown parameter  $[\delta_4 \ \sigma^2]^T$  *w.r.t.* the interference subspace  $\langle \mathbf{D} \rangle$  with known SOI can be linked to the non-centrality parameter, given in (64), according to

$$\lambda_4(P_{fa}, P_d) = \delta_4^2 \text{I-CRB}^{-1}(\delta_4). \quad (68)$$

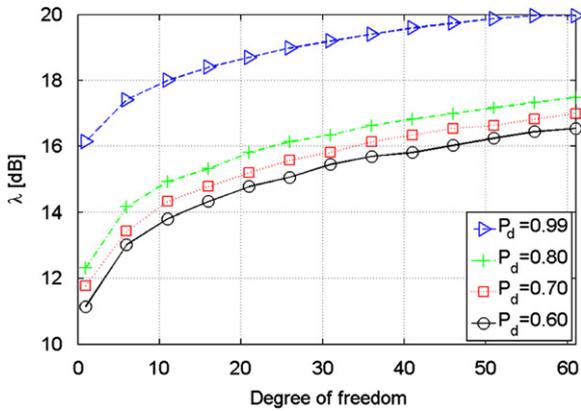
## 7. Summary of results and discussion

The studied cases are summarized in Table 1.<sup>3</sup>

<sup>3</sup> In this section,  $P_{fa}, P_d$  are dropped from  $\lambda(P_{fa}, P_d)$  for the sake of simplicity.

**Table 1**  
Summary of results with  $\mathbf{w} = \mathbf{B}\mathbf{s}_-$  and  $\mathbf{D} = [\mathbf{A} \ \mathbf{C}]$ .

	SOI	SI	Noise variance	The distribution used to compute $\lambda$	SINR for $M \geq 2$	SNR for $M=2$
Case 1	Known	Known	Known	$\chi_1^2$	$\frac{\ \mathbf{s}_1\ ^2 + \ \mathbf{s}_2\ ^2}{\ \mathbf{s}\ ^2 + \frac{2}{\lambda_1} \delta_1^2 \ \mathbf{w}\ ^2}$	$\lambda_1 \frac{\ \mathbf{s}_1\ ^2 + \ \mathbf{s}_2\ ^2}{2\delta_1^2 \ \mathbf{w}\ ^2}$
Case 2	Known	Unknown	Known	$\chi_1^2$	$\frac{\ \mathbf{s}_1\ ^2 + \ \mathbf{s}_2\ ^2}{\ \mathbf{s}\ ^2 + \frac{2}{\lambda_2} \delta_2^2 \ \mathbf{P}_C^\perp \mathbf{w}\ ^2}$	$\lambda_2 \frac{\ \mathbf{s}_1\ ^2 + \ \mathbf{s}_2\ ^2}{2\delta_2^2 \ \mathbf{w}\ ^2}$
Case 3	Unknown	Unknown	Known	$\chi_{2L}^2$	$\frac{\ \mathbf{s}_1\ ^2 + \ \mathbf{s}_2\ ^2}{\ \mathbf{s}\ ^2 + \frac{2}{\lambda_3} \delta_3^2 \ \mathbf{P}_D^\perp \mathbf{w}\ ^2}$	$\lambda_3 \frac{\ \mathbf{s}_1\ ^2 + \ \mathbf{s}_2\ ^2}{2\delta_3^2 \ \mathbf{P}_A^\perp \mathbf{w}\ ^2}$
Case 4	Unknown	Unknown	Unknown	$F_{2L, 2(N-M)}$	$\frac{\ \mathbf{s}_1\ ^2 + \ \mathbf{s}_2\ ^2}{\ \mathbf{s}\ ^2 + \frac{2}{\lambda_4} \delta_4^2 \ \mathbf{P}_D^\perp \mathbf{w}\ ^2}$	$\lambda_4 \frac{\ \mathbf{s}_1\ ^2 + \ \mathbf{s}_2\ ^2}{2\delta_4^2 \ \mathbf{P}_A^\perp \mathbf{w}\ ^2}$



**Fig. 2.** Behavior of the non-central parameter of  $\chi_n^2$  versus  $n$  the degree of freedom for different values of  $P_d$  and for a fixed  $P_{fa} = 0.01$ .

Toward the comparison of the studied cases, we formulate the three following propositions:

P1. First, as shown in Fig. 2, an increasing the number of the unknown parameter increases the degree of freedom of the  $\chi_n^2$  distribution used to derive  $\lambda$ , which will increase the value of  $\lambda$  and consequently,

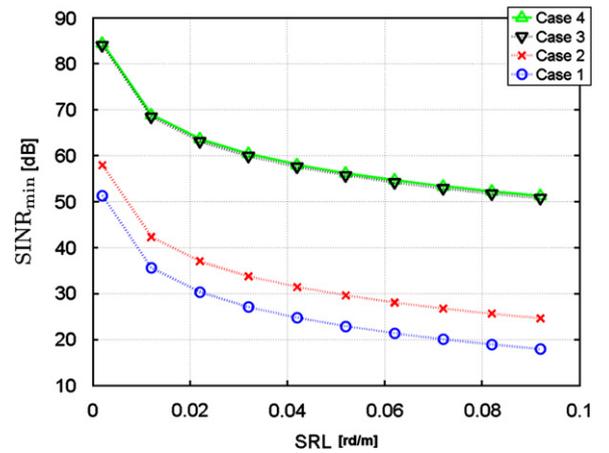
$$\lambda_1 = \lambda_2 \leq \lambda_3. \quad (69)$$

P2. Second, considering the noise variance as unknown parameter will produce a  $F$  distribution to compute the desired non-centrality parameter (see case 4). As a consequence, the non-centrality parameter computed *w.r.t.*  $\chi_n^2$  distribution is lower than the non-centrality parameter computed *w.r.t.*  $F$  distribution for any  $P_d > P_{fa}$  [26], meaning that

$$\lambda_3 \leq \lambda_4. \quad (70)$$

P3. On the other hand, note that  $\langle \mathbf{C} \rangle \subset \langle \mathbf{D} \rangle$ , where  $\langle \mathbf{C} \rangle$  and  $\langle \mathbf{D} \rangle$  denote the subspace spanned by the column of the matrices  $\mathbf{C}$  and  $\mathbf{D}$ , respectively. Consequently we have  $\forall \mathbf{w}: \mathbf{w}^H \mathbf{P}_C \mathbf{w} \leq \mathbf{w}^H \mathbf{P}_D \mathbf{w}$  and thus

$$\|\mathbf{P}_D^\perp \mathbf{w}\|^2 \leq \|\mathbf{P}_C^\perp \mathbf{w}\|^2 \leq \|\mathbf{w}\|^2. \quad (71)$$



**Fig. 3.**  $\text{SINR}_{\min}$  required to resolve two SOI *w.r.t.* the SRL for the studied cases.

From P1, P2 and P3 and for the same SRL (*i.e.*,  $\delta_1 = \delta_2 = \delta_3 = \delta_4$ ), one deduces that  $\text{SINR}_1 \leq \text{SINR}_2 \leq \text{SINR}_3 \leq \text{SINR}_4$ .

The same analysis can be done in the case of  $M=2$  (no interference), *i.e.*,

$$\text{SNR}_1 \leq \text{SNR}_2 \leq \text{SNR}_3 \leq \text{SNR}_4,$$

$$\text{SNR}_{10} \leq \text{SNR}_{20} \leq \text{SNR}_{30} \leq \text{SNR}_{40}.$$

In Fig. 3, we have reported the minimum SINR required to resolve two closely spaced SOI *w.r.t.* the SRL obtained in all cases. The gap between cases 1 and 2 is evaluated around 10 dB and it is especially due to the projector  $\mathbf{P}_C^\perp$ . The difference between cases 2 and 3 is around 25 dB. This loss is considerably high because it is due to the projector  $\mathbf{P}_D^\perp$  but also to the higher degree of freedom for  $\lambda_3$ . Finally, the gap between cases 3 and 4 is about only 0.5 dB and it is produced by the difference in the distribution used to compute the desired  $\lambda_4$ . In conclusion, the difference between the studied cases is mainly due to

- the non-centrality parameter numerical value,
- the effect of the subspace interference according to the projection onto  $\langle \mathbf{C} \rangle$  or  $\langle \mathbf{D} \rangle$ .

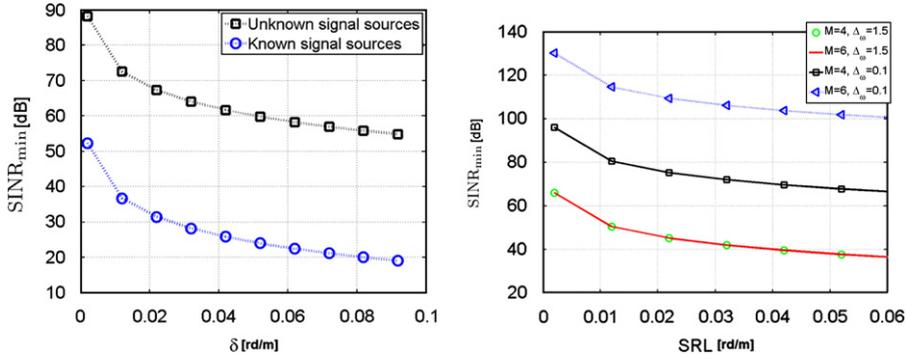


Fig. 4. (left) SINR to resolve two known/unknown closely far-field SOI with known noise variance for an ULA where  $N=10$ ,  $d = v/2$  and  $M=4$  in which  $\Delta\omega = 0.75$ . (right) The minimum SINR required to resolve two closely spaced SOI for an ULA where  $N=10$ ,  $d = v/2$ , for different number of sources and  $\Delta\omega$ .

8. Numerical analysis

This section is devoted to the numerical analysis of the minimum SINR required to resolve two closely spaced SOI w.r.t. the SRL. Furthermore, we have considered equal interference's power and broadband noise's power ( $INR = 1$ ) and thus  $SINR = \frac{1}{2} SNR$ . The number of snapshots is equal to  $L=100$  where  $v = 0.5$  m and  $(P_{fa}, P_d) = (0.01, 0.99)$ . The SRL w.r.t. the two closest sources (the SOI) is denoted by  $\delta$ , where all the remain sources are equally spaced by  $\Delta\omega$  (where  $\Delta\omega > \delta$ ).

8.1. Effect on the source prior

The prior knowledge on the source amplitudes and source phases is known to have a considerable effect on the estimation accuracy [27]. One could expect the same behavior concerning the resolution limit. From Fig. 4(left) one can notice the effect of the sources prior knowledge on the SRL. Indeed, the SRL depends strongly on the prior sources knowledge, e.g., the minimum SINR required to resolve two closely spaced known SOI w.r.t.  $\delta$  is approximately 40 dB less than the minimum SINR required to resolve two closely spaced unknown SOI.

8.2. Effect of the subspace interference

In Fig. 4(right), we have reported the effect of additional sources (considered as a subspace interference) on the minimum SINR required to resolve two closely spaced SOI. One can distinguish two cases:

1. The first one represents the scenario where  $\Delta\omega \gg \delta$ . In this case, one can notice that the additional sources do not affect the SINR. This can be explained by the fact that the high resolution algorithms have asymptotically an infinite resolving power [19].
2. The second scenario is for  $\Delta\omega > \delta$ . In this case, one can notice the drastic effect of the interfering sources. For example, the SINR gap between  $M=4$  and  $M=6$  scenarios is evaluated around 30 dB.

8.3. Orthogonal SOI

From an estimation point of view, it is well-known that the estimation accuracy for orthogonal signal sources

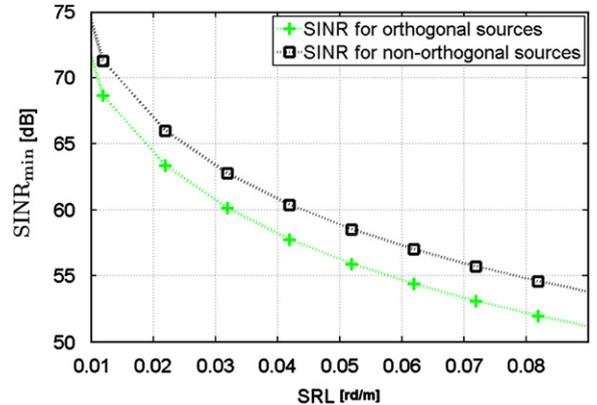


Fig. 5. The required SINR to resolve two BPSK unknown orthogonal/non-orthogonal closely far-field sources for an ULA where  $N=10$ ,  $d = v/2$  and  $M=4$ .

outperforms the estimation accuracy for the non-orthogonal signal sources [28]. One expect the same behavior concerning the minimum SINR required to resolve two closely spaced SOI. In fact, as shown in Fig. 5, the minimum SINR required to resolve two closely spaced SOI in the case of non-orthogonal binary phase-shift keying (BPSK) signal sources is greater than the case of orthogonal BPSK signal sources. This loss is around 3 dB.

8.4. Analysis for nonuniform arrays

The effect of the nonuniform antenna array is studied in the following. The linear array will be specified by their array aperture and their sensor positions.<sup>4</sup>

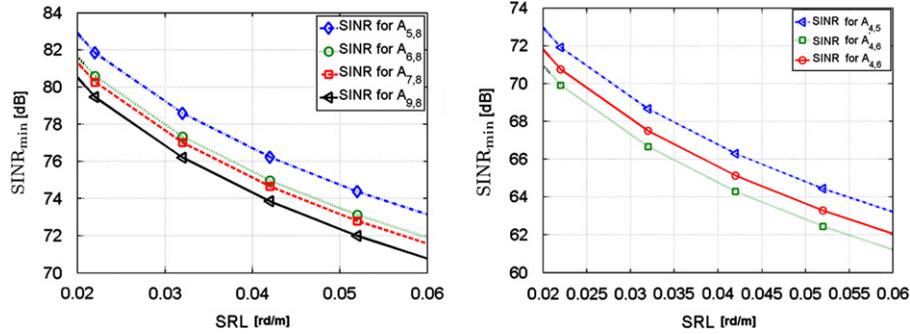
- First, let us study the effect of the number of sensors on the SRL (or, equivalently, on the minimum SINR required to resolve two closely spaced sources). In Table 2 are listed different array geometries with five, six, seven and nine sensors. The array with nine sensors is an ULA, whereas the others belong to the so-called

<sup>4</sup> For example, an ULA of  $N$  sensors will be represented as  $A_{N,N-1} = [0, 1, \dots, N-1]$ , where the subscript  $N-1$  is related to the array aperture (i.e., the distance between the first and the last sensor is equal to  $(N-1)d$  where  $d = v/2$  [29].)

**Table 2**

Characteristic of different array geometries with different number of sensors and with the same array aperture.

Array type	Sensor positions	$N$	Aperture	Redundant lags	Missing gaps
Minimum redundancy $A_{5,8}$	[0, 1, 2, 5, 8]	5	$8d$	$R = \{1, 3\}$	$G = \{\}$
Minimum redundancy $A_{6,8}$	[0, 1, 2, 3, 6, 8]	6	$8d$	$R = \{1, 2, 3, 5, 6\}$	$G = \{\}$
Minimum redundancy $A_{7,8}$	[0, 1, 2, 4, 5, 6, 8]	7	$8d$	$R = \{1, 2, 3, 4, 5, 6\}$	$G = \{\}$
ULA $A_{9,8}$	[0, 1, 2, 3, 4, 5, 6, 7, 8]	9	$8d$	$R = \{1, 2, 3, 4, 5, 6, 7\}$	$G = \{\}$



**Fig. 6.** (left) The required SINR to resolve two unknown closely spaced sources with known noise variance for different array geometries and same aperture which  $N=10$ ,  $d = v/2$  and  $M=4$  in which  $\Delta\omega = 1.5$ . (right) The required SNR to resolve two known sources using ULA, Type 4 and Type 5 geometries where  $P_{fa} = 0.01$  and  $P_d = 0.99$ .

**Table 3**

Characteristic of different array geometries with the same number of sensors and different array aperture. The so-called perfect array contains no redundancy lag and no gap.

Array type	Sensor positions	$N$	Aperture	Redundant lags	Missing gaps
Perfect array $A_{4,6}$	[0, 1, 4, 6]	4	$6d$	$R = \{\}$	$G = \{\}$
$A_{4,6}$	[0, 1, 2, 6]	4	$6d$	$R = \{1\}$	$G = \{3\}$
Minimum redundancy $A_{4,5}$	[0, 1, 2, 5]	4	$5d$	$R = \{1\}$	$G = \{\}$

“optimal” nonuniform array geometries [30]. More precisely, an exhaustive search has been done to select the minimum redundancy arrays with five, six and seven sensors with an aperture equals to  $8d$  (recall that the minimum redundancy arrays minimize the number of redundant lags  $R$  such that no missing lags will be present). From Fig. 6(left) one can notice, for the same array aperture, that the minimum SINR required to resolve two closely spaced SOI is slightly sensitive to the number of sensors. The gap for ULA of five sensors and the one for nine sensors (having the same array aperture) is evaluated at 2 dB.

- Finally, let us consider the case of different LA geometries with the same number of sensors. In Table 3 are reported different array geometries for  $N=4$  sensors with different apertures. One can notice, from Fig. 6(right), that the array aperture affects the minimum SINR required to resolve two closely spaced SOI is around 2 dB. On the other hand, one can notice that the SRL for arrays of the same aperture with different array geometries are affected by only 1 dB (i.e., between the so-called perfect array  $A_{4,6}$  and any array  $A_{4,6}$ ). Meaning that, the SRL is only slightly sensitive to the array design (for the same array aperture).

## 9. Conclusion

In this paper, we have linked theoretical expressions of the minimum signal-to-interference-plus-noise ratio (SINR) required to resolve two closely spaced far-field narrowband sources among a total number of  $M \geq 2$  impinging on a linear nonuniform array, and the statistical-resolution-limit (SRL). The two sources of interest (SOI) are corrupted by (1) the interference resulting from the  $M-2$  remaining sources and by (2) a broadband noise. Since our approach is based on the detection theory, these expressions provide useful information concerning the resolution limit for a given couple of probability of false alarm and probability of detection. In addition, the theoretical SINR required to resolve two SOI and the SRL have been analyzed with respect to the interference (resulting from the  $M-2$  other sources), the array geometry and the aperture, the *prior* sources knowledge or their orthogonality.

## Appendix A

### A.1. Derivation of the CMLE for cases 1 and 2

#### A.1.1. MLEs for case 1

The negative log-likelihood function is given by

$$L(\mathbf{z}, \delta) = -\ln p(\mathbf{z}) = -\ln(\pi\sigma^2)^{-NL/2} + \sigma^{-2}\|\mathbf{z} - \mathbf{w}\delta\|^2.$$

The optimization problem is given by

$$\arg \min_{\delta} L(\mathbf{z}, \delta) \quad \text{subject to } \delta \in \mathbb{R}.$$

This problem can be solved by the Lagrange multiplier method. Let  $\vartheta$  be a real Lagrange multiplier, then the Lagrange function is given by

$$L(\delta, \vartheta) = L(\mathbf{z}, \delta) + \vartheta \mathfrak{I}(\delta).$$

The condition  $\Im(\delta) = 0$  can be rewritten according to  $-j\frac{1}{2}(\delta - \delta^*) = 0$ . So, the partial derivatives of the Lagrange function are

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \delta} = \sigma^{-2}(\|\mathbf{w}\|^2 \delta^* - \mathbf{z}^H \mathbf{w}) - j\frac{\vartheta}{2}, \\ \frac{\partial \mathcal{L}}{\partial \vartheta} = \Im(\delta). \end{cases}$$

since  $\partial \delta^* / \partial \delta = 0$ . By letting  $\partial \mathcal{L} / \partial \delta|_{\delta_0} = 0$ , we have

$$\delta_0 = \frac{\mathbf{w}^H \mathbf{z}}{\|\mathbf{w}\|^2} - j\frac{\vartheta \sigma^2}{2\|\mathbf{w}\|^2}. \quad (72)$$

Setting  $\partial \mathcal{L} / \partial \vartheta|_{\vartheta_0} = 0$ , we have

$$\Im(\delta_0) = \Im\left\{\frac{\mathbf{w}^H \mathbf{z}}{\|\mathbf{w}\|^2}\right\} - \frac{\vartheta \sigma^2}{2\|\mathbf{w}\|^2} = 0.$$

Consequently, the Lagrange multiplier is given by

$$\vartheta_0 = \frac{2}{\sigma^2} \Im\{\mathbf{w}^H \mathbf{z}\}.$$

Plugging the above expression in (72), we have

$$\hat{\delta} = \frac{\mathbf{w}^H \mathbf{z}}{\|\mathbf{w}\|^2} - j\Im\left\{\frac{\mathbf{w}^H \mathbf{z}}{\|\mathbf{w}\|^2}\right\} = \frac{\Re\{\mathbf{w}^H \mathbf{z}\}}{\|\mathbf{w}\|^2}, \quad (73)$$

by using  $\Re\{a\} = a - j\Im\{a\}$ .

#### A.1.2. MLEs for case 2

The negative log-likelihood function is given by

$$L(\mathbf{z}, \mathbf{d}) = -\ln p(\mathbf{z}) = -\ln(\pi\sigma^2)^{-NL/2} + \sigma^{-2}\|\mathbf{z} - \mathbf{Q}\mathbf{p}\|^2.$$

The optimization problem is given by

$$\arg \min_{\mathbf{p}} L(\mathbf{z}, \mathbf{p}) \quad \text{subject to } \mathbf{e}_1^T \mathbf{p} \in \mathbb{R},$$

where  $\mathbf{e}_1 = [1 \ 0 \ \dots \ 0]^T$ .

This problem can be solved by the Lagrange multiplier method. Let  $\vartheta$  be a real Lagrange multiplier, then the Lagrange function is given by

$$\mathcal{L}(\mathbf{p}, \vartheta) = L(\mathbf{z}, \mathbf{p}) + \vartheta \Im(\mathbf{e}_1^T \mathbf{p}).$$

The condition  $\Im(\mathbf{e}_1^T \mathbf{p}) = 0$  can be rewritten according to  $-j\frac{1}{2}(\mathbf{e}_1^T \mathbf{p} - \mathbf{e}_1^T \mathbf{p}^*) = 0$ . So, the partial derivatives of the Lagrange function are

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \mathbf{p}} = \sigma^{-2}(\mathbf{Q}^T \mathbf{Q}^* \mathbf{p}^* - \mathbf{Q}^T \mathbf{z}^*) - j\frac{\vartheta}{2} \mathbf{e}_1, \\ \frac{\partial \mathcal{L}}{\partial \vartheta} = \Im(\mathbf{e}_1^T \mathbf{p}), \end{cases}$$

since  $\partial \mathbf{p}^* / \partial \mathbf{p} = \mathbf{0}$ . By letting  $\partial \mathcal{L} / \partial \mathbf{p}|_{\mathbf{p}_0} = 0$ , we have

$$\mathbf{p}_0 = \mathbf{Q}^\dagger \mathbf{z} - j\frac{\vartheta \sigma^2}{2} (\mathbf{Q}^H \mathbf{Q})^{-1} \mathbf{e}_1. \quad (74)$$

By setting  $\partial \mathcal{L} / \partial \vartheta|_{\vartheta_0} = 0$ , we have

$$\Im(\mathbf{e}_1^T \mathbf{p}_0) = \Im(\mathbf{e}_1^T \mathbf{Q}^\dagger \mathbf{z}) - \frac{\vartheta_0 \sigma^2}{2} h = 0,$$

where we have defined the real quantity  $h = \mathbf{e}_1^T (\mathbf{Q}^H \mathbf{Q})^{-1} \mathbf{e}_1$ . Consequently, the Lagrange multiplier is given by

$$\vartheta_0 = \frac{2}{\sigma^2 h} \Im(\mathbf{e}_1^T \mathbf{Q}^\dagger \mathbf{z}).$$

Plugging the above expression into (74), we have

$$\hat{\mathbf{p}} = \mathbf{Q}^\dagger \mathbf{z} - j\frac{1}{h} (\mathbf{Q}^H \mathbf{Q})^{-1} \mathbf{e}_1 \Im(\mathbf{e}_1^T \mathbf{Q}^\dagger \mathbf{z}). \quad (75)$$

*CMLE of the SRL:* The estimate of the SRL is given by  $\hat{\delta} = \mathbf{e}_1^T \hat{\mathbf{p}}$  and thus,

$$\hat{\delta} = \mathbf{e}_1^T \mathbf{Q}^\dagger \mathbf{z} - j\Im(\mathbf{e}_1^T \mathbf{Q}^\dagger \mathbf{z}). \quad (76)$$

Now, remark that  $\Re\{a\} = a - j\Im\{a\}$ , then

$$\hat{\delta} = \Re\{\mathbf{e}_1^T \mathbf{Q}^\dagger \mathbf{z}\}. \quad (77)$$

In addition, using the inverse of a block matrix and the Schur complement [10], we have

$$\mathbf{e}_1^T \mathbf{Q}^\dagger = \ell \mathbf{w}^H + \mathbf{u}^H \mathbf{C}^H, \quad (78)$$

where

$$\ell = \frac{1}{\|\mathbf{w}\|^2 - \mathbf{w}^H \mathbf{C} (\mathbf{C}^H \mathbf{C})^{-1} \mathbf{C}^H \mathbf{w}} = \frac{1}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2}, \quad (79)$$

$$\mathbf{u}^H = -\frac{\mathbf{w}^H \mathbf{C} (\mathbf{C}^H \mathbf{C})^{-1}}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2}. \quad (80)$$

Thus

$$\mathbf{e}_1^T \mathbf{Q}^\dagger = \frac{\mathbf{w}^H \mathbf{P}_C^\perp}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2}. \quad (81)$$

Using (77) and (81), we have (26).

*MLE of the interfering sources:* The estimate of the interfering sources is given by  $\hat{\mathbf{s}} = \mathbf{J} \hat{\mathbf{p}}$  where  $\mathbf{J} = [\mathbf{0}_{(M-2)L \times 1} \ \mathbf{I}_{(M-2)L}]$  is an  $((M-2)L) \times ((M-2)L + 1)$  selection matrix. We have

$$\hat{\mathbf{s}} = \mathbf{J} \mathbf{Q}^\dagger \mathbf{z} - j\frac{1}{h} \mathbf{J} (\mathbf{Q}^H \mathbf{Q})^{-1} \mathbf{e}_1 \Im(\mathbf{e}_1^T \mathbf{Q}^\dagger \mathbf{z}). \quad (82)$$

Let us define the following matrix:

$$\mathbf{G} \triangleq (\mathbf{C}^H \mathbf{C})^{-1} \left( \mathbf{I} + \frac{\mathbf{C}^H \mathbf{w} \mathbf{w}^H \mathbf{C}^H}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2} \right), \quad (83)$$

and observe the following equalities:

$$\mathbf{J} \mathbf{Q}^\dagger = \mathbf{u} \mathbf{w}^H + \mathbf{G} \mathbf{C}^H = \mathbf{C}^\dagger - \frac{1}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2} \mathbf{C}^\dagger \mathbf{w} \mathbf{w}^H \mathbf{P}_C^\perp, \quad (84)$$

$$\frac{1}{h} \mathbf{J} (\mathbf{Q}^H \mathbf{Q})^{-1} \mathbf{e}_1 = \frac{1}{h} [\mathbf{u} \ \mathbf{G}] \mathbf{e}_1 = \frac{1}{h} \mathbf{u} = -\mathbf{C}^\dagger \mathbf{w}. \quad (85)$$

Plugging the two above expressions and (81) into (82), we obtain

$$\hat{\mathbf{s}} = \mathbf{C}^\dagger \mathbf{z} - \frac{\mathbf{C}^\dagger \mathbf{w} \mathbf{w}^H \mathbf{P}_C^\perp}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2} \mathbf{z} + j \mathbf{C}^\dagger \mathbf{w} \Im\left\{\frac{\mathbf{w}^H \mathbf{P}_C^\perp \mathbf{z}}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2}\right\} \quad (86)$$

$$\hat{\mathbf{s}} = \mathbf{C}^\dagger \mathbf{z} - \underbrace{\mathbf{C}^\dagger \mathbf{w} \left( \frac{\mathbf{w}^H \mathbf{P}_C^\perp \mathbf{z}}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2} - j \Im\left\{\frac{\mathbf{w}^H \mathbf{P}_C^\perp \mathbf{z}}{\|\mathbf{P}_C^\perp \mathbf{w}\|^2}\right\} \right)}_{\hat{\delta}} \quad (87)$$

$$\hat{\mathbf{s}} = \mathbf{C}^\dagger (\mathbf{z} - \mathbf{w} \hat{\delta}). \quad (88)$$

#### A.2. Statistic of the random variable $\Re^2\{\mathbf{w}^H \mathbf{y}\} / (\sigma^2/2)\|\mathbf{w}\|^2$

Let us consider a random variable  $\mathbf{y} = \delta \mathbf{w} + \mathbf{v}$  corrupted by a zero-mean white circular Gaussian noise  $\mathbf{v}$  of

variance  $\sigma^2$ . We recall that a circular random variable means [22]  $\Re\{\mathbf{v}\} \sim \mathcal{N}(\mathbf{0}, (\sigma^2/2)\mathbf{I})$ ,  $\Im\{\mathbf{v}\} \sim \mathcal{N}(\mathbf{0}, (\sigma^2/2)\mathbf{I})$  and  $E\{\Re\{\mathbf{v}\}\Im\{\mathbf{v}\}^T\} = E\{\Im\{\mathbf{v}\}\Re\{\mathbf{v}\}^T\} = \mathbf{0}$ . So,

$$\begin{cases} \mathbf{y}_{\mathcal{H}_0} \sim \mathcal{CN}(\mathbf{0}, \sigma^2\mathbf{I}), \\ \mathbf{y}_{\mathcal{H}_1} \sim \mathcal{CN}(\delta\mathbf{w}, \sigma^2\mathbf{I}). \end{cases} \quad (89)$$

Let  $u = \Re\{\mathbf{w}^H\mathbf{y}\}$ . The mean of variable  $u$  is given by  $\|\mathbf{w}\|^2\delta$  and its variance is

$$\begin{aligned} C_u &= E\{(\Re\{\mathbf{w}^H\mathbf{y}\} - \|\mathbf{w}\|^2\delta)^2\} = E\{\Re^2\{\mathbf{w}^H\mathbf{v}\}\} \\ &= E\{(\Re\{\mathbf{w}^H\}\Re\{\mathbf{v}\} - \Im\{\mathbf{w}^H\}\Im\{\mathbf{v}\})^2\} \\ &= E\{(\Re\{\mathbf{w}\}^T\Re\{\mathbf{v}\} + \Im\{\mathbf{w}\}^T\Im\{\mathbf{v}\})^2\}. \end{aligned}$$

Consequently, using the circularity of the noise, one obtains

$$C_u = \Re\{\omega\}^T \underbrace{E\{\Re\{\mathbf{v}\}\Re\{\mathbf{v}\}^T\}}_{(\sigma^2/2)\mathbf{I}} \Re\{\omega\} + \Im\{\omega\}^T \underbrace{E\{\Im\{\mathbf{v}\}\Im\{\mathbf{v}\}^T\}}_{(\sigma^2/2)\mathbf{I}} \Im\{\omega\} \quad (90)$$

$$+ \underbrace{\Re\{\omega\}^T E\{\Re\{\mathbf{v}\}\Im\{\mathbf{v}\}^T\}}_{\mathbf{0}} \Im\{\omega\} + \underbrace{\Im\{\omega\}^T E\{\Im\{\mathbf{v}\}\Re\{\mathbf{v}\}^T\}}_{\mathbf{0}} \Re\{\omega\}, \quad (91)$$

$$C_u = \frac{\sigma^2}{2} \|\mathbf{w}\|^2. \quad (92)$$

Let us define a new statistic as follows:

$$T(\mathbf{y}) \triangleq \frac{u^2}{C_u} = \frac{\Re^2\{\mathbf{w}^H\mathbf{y}\}}{\frac{\sigma^2}{2}\|\mathbf{w}\|^2}. \quad (93)$$

Thus, according to [15], we have  $T(\mathbf{y}) \sim \chi^2_2(\lambda)$  in which  $\chi^2_2(\lambda)$  denotes the non-central chi-square distribution with one degree of freedom where the non-centrality parameter is given by

$$\lambda \triangleq \frac{E(u)^2}{C_u} = \frac{2\delta^2\|\mathbf{w}\|^2}{\sigma^2}. \quad (94)$$

### A.3. Derivation of the CRB and the I-CRB

In this appendix, we derive the CRB (Cramér-Rao Bound) and the so-called I-CRB (interference CRB) [23]. Let  $E\{(\hat{\Theta} - \Theta)(\hat{\Theta} - \Theta)^T\}$  be the covariance matrix of an unbiased estimator,  $\hat{\Theta}$ , of the deterministic parameter vector  $\Theta$ . The covariance inequality principle states that, under quite general/weak conditions, the variance satisfies:  $\text{MSE}([\hat{\Theta}]_i) = E\{([\hat{\Theta}]_i - [\Theta]_i)^2\} \geq [\mathbf{CRB}(\Theta)]_{i,i}$  where  $\mathbf{CRB}(\Theta) = \mathbf{FIM}^{-1}(\Theta)$ , in which  $\mathbf{FIM}$  denotes the Fisher Information Matrix. The  $(i, k)$ th element of the FIM for the parameter vector  $\Theta$  can be written (for a complex circular Gaussian observation model) as [22]

$$\begin{aligned} [\mathbf{FIM}(\Theta)]_{i,k} &= \text{tr} \left\{ \mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial [\Theta]_i} \mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial [\Theta]_k} \right\} \\ &+ 2\Re \left\{ \frac{\partial \mu^H}{\partial [\Theta]_i} \mathbf{R}^{-1} \frac{\partial \mu}{\partial [\Theta]_k} \right\}, \end{aligned} \quad (95)$$

where  $\mathbf{R}$  and  $\mu$  denote the covariance matrix and the mean of the observation vector model, respectively.

#### A.3.1. Derivation of the CRB

Let us consider the estimation of the real parameter of interest  $\delta$ , where the observation model is as follows:

$$\mathbf{z} = \delta\mathbf{w} + \mathbf{v}, \quad (96)$$

where  $\mathbf{v} \sim \mathcal{CN}(0, \sigma^2\mathbf{I})$  whereas  $\delta$ ,  $\mathbf{w}$  and  $\sigma^2$  are deterministic parameters. Thus  $\mathbf{z} \sim \mathcal{CN}(\mu = \delta\mathbf{w}, \mathbf{R} = \sigma^2\mathbf{I})$ . The unknown deterministic parameter vector is defined as  $\Theta = [\delta \ \sigma^2]^T$ . Using (95), the CRB w.r.t.  $\delta$  for the observation (96) is given by

$$\text{CRB}(\delta) = \left[ \frac{2}{\sigma^2} \Re \left\{ \left( \frac{\partial \mu}{\partial \delta} \right)^H \frac{\partial \mu}{\partial \delta} \right\} \right]^{-1} = \frac{\sigma^2}{2} \|\mathbf{w}\|^2 \quad (97)$$

since it is well-known that  $\delta$  and  $\sigma^2$  are decoupled (diagonal FIM).

#### A.3.2. Derivation of the I-CRB

Now, let us consider the estimation of the real parameter of interest  $\delta$ , where the observation model is corrupted by a deterministic structured interference as follows:

$$\mathbf{z} = \delta\mathbf{w} + \mathbf{C}\mathbf{s} + \mathbf{v}. \quad (98)$$

Let us define the orthogonal projector and its orthogonal decomposition according to  $\mathbf{P}_C^\perp = \mathbf{U}\mathbf{U}^H$  which is a null-steering operator that nulls everything in the interference space  $\langle \mathbf{C} \rangle$  [24]. Let us define a new observation based on (98) as follows:

$$\tilde{\mathbf{z}} \triangleq \mathbf{U}^H \mathbf{z} = \delta \mathbf{U}^H \mathbf{w} + \tilde{\mathbf{v}}, \quad (99)$$

since  $\mathbf{U}^H \mathbf{U} = \mathbf{I}$ , one has  $\tilde{\mathbf{v}} = \mathbf{U}^H \mathbf{v} \sim \mathcal{CN}(0, \sigma^2\mathbf{I})$  and  $\mathbf{z} \sim \mathcal{CN}(\mu = \delta \mathbf{U}^H \mathbf{w}, \mathbf{R} = \sigma^2\mathbf{I})$ . The I-CRB [23] is the CRB for the observation (99) related to the projector  $\mathbf{P}_C^\perp$  where the unknown vector parameter is given by  $\Theta$ . Consequently, using (95) and after straightforward calculus, one obtains

$$\text{I-CRB}(\delta) = \left[ \frac{2}{\sigma^2} \Re \left\{ \left( \frac{\partial \mu}{\partial \delta} \right)^H \frac{\partial \mu}{\partial \delta} \right\} \right]^{-1} = \frac{\sigma^2}{2} \|\mathbf{P}_C^\perp \mathbf{w}\|^2 \quad (100)$$

since it is well-known that  $\delta$  and  $\sigma^2$  are decoupled (diagonal FIM).

#### A.4. Independence of $\|\bar{\mathbf{y}}\|^2$ and $\|\hat{\mathbf{y}}\|^2$

Since  $E(\bar{\mathbf{y}}) = \mathbf{0}$  under  $\mathcal{H}_0$  and  $\mathcal{H}_1$ , one has

$$\begin{aligned} \text{Cov}(\bar{\mathbf{y}}, \hat{\mathbf{y}}) &= E(\bar{\mathbf{y}}\hat{\mathbf{y}}^H) = \mathbf{U}^H E(\mathbf{y}\mathbf{y}^H) \mathbf{U} \\ &= \mathbf{U}^H \mathbf{U}^H \mathbf{U}^H E(\mathbf{y}\mathbf{y}^H) \mathbf{U} \mathbf{U}^H \mathbf{U} = \mathbf{U}^H \mathbf{P}_{[\text{BD}]}^\perp E(\mathbf{y}\mathbf{y}^H) \mathbf{P}_{\mathbf{P}_D^\perp \mathbf{B}} \mathbf{U} \\ &= \mathbf{U}^H (\sigma^2 \mathbf{P}_{[\text{BD}]}^\perp \mathbf{P}_{\mathbf{P}_D^\perp \mathbf{B}} + (\mathbf{P}_{[\text{BD}]}^\perp \mathbf{e})(\mathbf{P}_{\mathbf{P}_D^\perp \mathbf{B}} \mathbf{e})^H) \mathbf{U}, \end{aligned}$$

where  $\mathbf{e} = \mathbf{B}\theta + \mathbf{D}\mathbf{g}$  under  $\mathcal{H}_1$  and  $\mathbf{e} = \mathbf{D}\mathbf{g}$  under  $\mathcal{H}_0$ .

Note  $\mathbf{P}_{[\text{BD}]}^\perp \mathbf{e} = \mathbf{0}$ . And, on the other hand,

$$\begin{aligned} \mathbf{P}_{[\text{BD}]}^\perp \mathbf{P}_{\mathbf{P}_D^\perp \mathbf{B}} &= \mathbf{P}_D^\perp (\mathbf{P}_D^\perp \mathbf{E}_{\text{BD}} - \mathbf{E}_{\text{BD}} \mathbf{P}_D^\perp \mathbf{E}_{\text{BD}}) \mathbf{P}_D^\perp \\ &= (\mathbf{P}_D^\perp \mathbf{E}_{\text{BD}} - \mathbf{P}_D^\perp \mathbf{E}_{\text{BD}} \mathbf{P}_D^\perp \mathbf{E}_{\text{BD}}) \mathbf{P}_D^\perp \\ &= (\mathbf{P}_D^\perp \mathbf{E}_{\text{BD}} - \mathbf{P}_D^\perp \mathbf{E}_{\text{BD}}) \mathbf{P}_D^\perp = \mathbf{0}. \end{aligned}$$

Consequently,  $\text{Cov}(\bar{\mathbf{y}}, \hat{\mathbf{y}}) = \mathbf{0}$ . Meaning that  $\bar{\mathbf{y}}$  and  $\hat{\mathbf{y}}$  are uncorrelated. Thus, they are independent in the normal distribution case [25]. Consequently, it is straightforward to conclude that  $\|\bar{\mathbf{y}}\|^2$  and  $\|\hat{\mathbf{y}}\|^2$  are also independent [16].

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