# A Fresh Look at the Bayesian Bounds of the Weiss-Weinstein Family

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**Abstract**

Minimal bounds on the mean square error are generally used in order to predict the best achievable performance of an estimator for a given observation model. In this paper we are interested in the Bayesian bound of the Weiss-Weinstein family. Among this family, we have Bayesian Cramér-Rao bound, the Bobrovsky-MayerWolf-Zakaï bound, the Bayesian Bhattacharyya bound, the Bobrovsky-Zakaï bound, the Reuven-Messer bound, and the Weiss-Weinstein bound. We present a unification of all these minimal bounds based on a rewriting of the minimum mean square error estimator and on a constrained optimization problem. With this approach, we obtain a useful theoretical framework to derive new Bayesian bounds. For that purpose, we propose two bounds. First, we propose a generalization of the Bayesian Bhattacharyya bound extending the works of Bobrovsky, Mayer-Wolf, and Zakaï. Second, we propose a bound based on the Bayesian Bhattacharyya bound and on the Reuven-Messer bound, representing a generalization of these bounds. The proposed bound is the Bayesian extension of the deterministic Abel bound and is found to be tighter than the Bayesian Bhattacharyya bound, the Reuven-Messer bound, the Bobrovsky-Zakaï bound, and the Bayesian Cramér-Rao bound. We propose some closed-form expressions of these bounds for a general Gaussian observation model with parameterized mean. In order to illustrate our results, we present simulation results in the context of a spectral analysis problem.

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## NOTATIONS

The notational convention adopted in this paper is as follows: italic indicates a scalar quantity, as in $A$; lowercase boldface indicates a vector quantity, as in $\mathbf{a}$; uppercase boldface indicates a matrix quantity, as in $\mathbf{A}$. $\mathrm{Re}\{A\}$ is the real part of $A$ and $\mathrm{Im}\{A\}$ is the imaginary part of $A$. The complex conjugation of a quantity is indicated by a superscript * as in $A^*$. The matrix transpose is indicated by a superscript $^T$ as in $\mathbf{A}^T$, and the complex conjugate plus matrix transpose is indicated by a superscript $^H$ as in $\mathbf{A}^H = \left(\mathbf{A}^T\right)^*$. The *n*-th row and *m*-th column element of the matrix $\mathbf{A}$ is denoted by $\{\mathbf{A}\}_{n,m}$. $\mathbf{I}_N$ denotes the identity matrix of size $N \times N$. $\mathbf{0}_{N\times M}$ is a $N \times M$ matrix of zeros. $\|.\|$ denotes the norm. $|.|$ denotes the modulus. $abs\,(.)$ denotes the absolute value. $\delta\,(.)$ denotes the Dirac delta function. $\mathbb{E}\,[.]$ denotes the expectation operator with respect to a density probability function explicitly given by a subscript. $\Omega$ is the observation space and $\Theta$ the parameter space.

## I. INTRODUCTION

Minimal bounds on the Mean Square Error (MSE) provide the ultimate performance that an estimator can expect to achieve for a given observation model. Consequently, they are used as a benchmark in order to evaluate the performance of an estimator and to determine if an improvement is possible. The Cramér-Rao bound [3]–[8] has been the most widely used by the signal processing community and is still under investigation from a theoretical point of view (particularly throughout the differential variety in the Riemannian geometry framework [9]–[14]) as from a practical point of view (see, e.g., [15]–[19]). But the Cramér-Rao bound suffers from some drawbacks when the scenario becomes critical. Indeed, in a non-linear estimation problem, when the parameters have finite support, there are three distinct MSE areas for an estimator [20] page 273, [21]. For a large number of observations or for a high Signal-to-Noise Ratio (SNR), the estimator MSE is small and the area is called an asymptotic area. When the scenario becomes critical, i.e., when the number of observations or the SNR decreases, the estimator MSE increases dramatically due to the outlier effect, and the area is called threshold area. Finally, when the number of observations or the SNR is low, the estimator criterion is hugely corrupted by the noise and becomes a quasi-uniform random variable on the parameter support. Since in this last area the observations bring almost no information, it is called no information area. The Cramér-Rao bound is used only in the asymptotic area and is not able to handle the threshold phenomena (i.e., when the performance breaks down).

To fill this lack, a plethora of other minimal bounds tighter than the Cramér-Rao bound has been proposed and studied. All these bounds have been derived by way of several inequalities, such as the Cauchy-Schwartz inequality, the Kotelnikov inequality, the Hölder inequality, the Ibragimov-Hasminskii inequality, the Bhattacharyya inequality and the Kiefer inequality. Note that due to this diversity, it is sometimes difficult to fully understand the underlying concept and the difference between all these bounds; consequently it is difficult to apply these bounds to a specific estimation problem.

Minimal bounds on the MSE can be divided into two categories: the deterministic bounds for situations in which the true vector of the parameters $\boldsymbol{\theta}_0$ is assumed to be deterministic and the Bayesian bounds for situations in which the vector of parameters $\boldsymbol{\theta}$ is assumed to be random with an *a priori* probability density function $p\,(\boldsymbol{\theta})$. Among the deterministic bounds, we have the well-known Cramér-Rao bound; the Bhattacharyya bound [22], [23]; the Chapman-Robbins bound [24]–[26], the Barankin bound [27], [28], the Abel bound [29]–[31]; and the Quinlan-Chaumette-Larzabal bound [32]. Bayesian bounds can be subdivided into two categories: the Ziv-Zakaï family, derived from a binary hypothesis testing problem (and more generally from an $M$-ary hypothesis testing problem), and the Weiss-Weinstein family, derived (as the deterministic bounds) from a covariance inequality principle. The Ziv-Zakaï family contains the Ziv-Zakaï bound [33], the Bellini-Tartara bound [34], the Chazan-Zakaï-Ziv bound [35], the Weinstein bound [36], the Bell-Steinberg-Ephraim-VanTrees bound [37], and the Bell bound [38]. The Weiss-Weinstein family contains the Bayesian Cramér-Rao bound [20] page 72, and 84, the Bobrovsky-MayerWolf-Zakaï bound [39], the Bayesian Bhattacharyya bound [20] page 149, the Bobrovsky-Zakaï bound [40], the Reuven-Messer bound [41], and the Weiss-Weinstein bound [42]. A nice tutorial about both families can be found in the recent book of Van Trees and Bell [43].

The deterministic bounds are used as a lower bound of the *local* MSE in $\boldsymbol{\theta}_0$; i.e.,

$$\mathbf{MSE}_L\left(\boldsymbol{\theta}_0\right) = \int\limits_{\Omega} \left(\hat{\boldsymbol{\theta}}\left(\mathbf{y}\right) - \boldsymbol{\theta}_0\right)\left(\hat{\boldsymbol{\theta}}\left(\mathbf{y}\right) - \boldsymbol{\theta}_0\right)^T p\left(\mathbf{y}|\,\boldsymbol{\theta}_0\right) d\mathbf{y}, \tag{1}$$

where $\mathbf{y} \in \Omega$ is a complex observation vector, $p\left(\mathbf{y}|\,\boldsymbol{\theta}_0\right)$ is the likelihood of the observations parameterized by the true parameter value $\boldsymbol{\theta}_0$, and $\hat{\boldsymbol{\theta}}\left(\mathbf{y}\right)$ is an estimator of $\boldsymbol{\theta}_0$.

On the other hand, Bayesian bounds are used as a lower bound of the *global* MSE; i.e.,

$$\mathbf{MSE}_G = \int\limits_{\Theta}\int\limits_{\Omega} \left(\hat{\boldsymbol{\theta}}\left(\mathbf{y}\right) - \boldsymbol{\theta}\right)\left(\hat{\boldsymbol{\theta}}\left(\mathbf{y}\right) - \boldsymbol{\theta}\right)^T p\left(\mathbf{y}, \boldsymbol{\theta}\right) d\mathbf{y} d\boldsymbol{\theta}, \tag{2}$$

where $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ is the random parameter vector with an *a priori* probability density function $p\left(\boldsymbol{\theta}\right) = \frac{p(\mathbf{y},\boldsymbol{\theta})}{p(\,\mathbf{y}|\boldsymbol{\theta})}$ and $p\left(\mathbf{y}, \boldsymbol{\theta}\right)$ is the joint probability function of the observations and of the parameters.

In the deterministic context, minimal bounds—in particular the Chapman-Robbins bound and the Barankin bound—are generally used in order to predict the aforementioned threshold effect which cannot be handled by the Cramér-Rao bound. The Chapman-Robbins bound and the Barankin bound have already been successfully applied to several estimation problems [28], [31], [44]–[55]. The use of the Abel bound, which can also handle the threshold phenomena, is still marginal [56].

Contrary to the deterministic bounds, the Bayesian bounds take into account the parameter support throughout the *a priori* probability density function $p\left(\boldsymbol{\theta}\right)$, and they give the ultimate performances of an estimator on the three aforementioned areas of the global MSE. These bounds give the performance of the Bayesian estimator, such as the Maximum a Posteriori (MAP) estimator or the Minimum Mean Square Error Estimator (MMSEE), and can be used in order to know the global performance of the deterministic estimators such as the Maximum Likelihood estimator (MLE), since

$$\mathbf{MSE}_G = \int\limits_{\Theta} \mathbf{MSE}_L\left(\boldsymbol{\theta}\right) p\left(\boldsymbol{\theta}\right) d\boldsymbol{\theta}. \tag{3}$$

The reader is referred to Xu *et al.* [57]–[59], where the MLE performances are analyzed in the context of an underwater acoustic problem by way of the Ziv-Zakaï and of the Weiss-Weinstein bounds. The Ziv-Zakaï family bounds have been applied in other signal processing areas: time-delay estimation [60]; direction-of-arrival estimation [38], [61], [62]; and digital communication [63]. On the other hand, the Weiss-Weinstein bound has been less investigated: the aforementioned Xu *et al.* works and in the framework of digital communication [64].

This article presents a new unified approach for the establishment of the Weiss-Weinstein family bounds. Note that the unification of the deterministic bounds has already been proposed by [65] and [66] based on a constrained optimization problem. A unification has been proposed by Bell *et al.* in [37], [38] for the Ziv-Zakaï family.

Concerning the Weiss-Weinstein family unification, a first approach has been given by Weiss and Weinstein in [67]. This approach is based on the following inequality proved by the authors:

$$MSE_G \geq \frac{\mathbb{E}^2_{\mathbf{y},\theta}\left[\theta\psi\left(\mathbf{y},\theta\right)\right]}{\mathbb{E}_{\mathbf{y},\theta}\left[\psi^2\left(\mathbf{y},\theta\right)\right]}, \tag{4}$$

where the function $\psi\left(\mathbf{y},\theta\right)$ must satisfied

$$\int_{\Theta}\psi\left(\mathbf{y},\theta\right)p\left(\mathbf{y},\theta\right)d\theta = 0. \tag{5}$$

Weiss and Weinstein gave several functions $\psi\left(\mathbf{y},\theta\right)$ satisfying (5) for which they again obtain the Bayesian Cramér-Rao bound, the Bayesian Bhattacharyya bound, the Bobrovsky-Zakaï bound, and the Weiss-Weinstein bound. Moreover, a function $\psi\left(\mathbf{y},\theta\right)$ satisfying (5) leading to the Bobrovsky-MayerWolf-Zakaï bound is given in [38]. Unfortunately, there are no general rules to find $\psi\left(\mathbf{y},\theta\right)$. In this contribution, the Weiss-Weinstein family unification is based on the best Bayesian bound, i.e., the MSE of the MMSEE. By rewriting the MMSEE and by using a constrained optimization problem similar to one derived for the unification of deterministic bounds [65], [66], we unify the Bayesian Cramér-Rao bound, the Bobrovsky-MayerWolf-Zakaï bound, the Bayesian Bhattacharyya bound, the Bobrovsky-Zakaï bound, the Reuven-Messer bound (for which no function $\psi\left(\mathbf{y},\theta\right)$ is proposed in the Weiss-Weinstein approach), and the Weiss-Weinstein bound. This approach brings a useful theoretical framework to derive new Bayesian bounds.

For that purpose, we propose two bounds. First, we propose a generalization of the Bayesian Bhattacharyya bound extending the works of Bobrovsky, Mayer-Wolf, and Zakaï. Second, we propose a bound based on the Bayesian Bhattacharyya bound and on the Reuven-Messer bound, one that represents a generalization of these bounds. This bound is found to be tighter than the Bayesian Bhattacharyya bound, the Reuven-Messer bound, the Bobrovsky-Zakaï bound, and the Bayesian Cramér-Rao bound. In order to illustrate our results, we propose some closed-form expressions of the minimal bounds for a Gaussian observation model with parameterized mean widely used in signal processing, and we apply it to a spectral analysis problem for which we present simulation results.

## II. Minimum MSE reformulation

In this section we start by reformulating the MMSEE as a constrained optimization problem. Then, we rewrite the underlying constraint under three different forms that will be of interest for our proposed unification.

In the Bayesian framework, the minimal global MSE and consequently the best Bayesian bound is the MSE of the MMSEE: $\hat{\theta}\left(\mathbf{y}\right) = \int_{\Theta}\theta p\left(\theta|\,\mathbf{y}\right)d\theta$, where $p\left(\theta|\,\mathbf{y}\right)$ is the *a posteriori* probability density function of the parameter.

Unfortunately, it is generally impossible to obtain a closed-form expression of this MSE. The MMSEE is the solution of the following problem:

$$\min_{\hat{\theta}(\mathbf{y})} \int_{\Theta}\int_{\Omega} \left(\hat{\theta}(\mathbf{y}) - \theta\right)^2 p(\mathbf{y},\theta)\, d\mathbf{y} d\theta. \tag{6}$$

Let $\mathcal{L}_p^2$ be the set of function $v(\mathbf{y},\theta)$ such that $\int_{\Theta}\int_{\Omega} v^2(\mathbf{y},\theta)\, p(\mathbf{y},\theta)\, d\mathbf{y} d\theta$ is defined. Let $\mathcal{C}_1 \subset \mathcal{L}_p^2$ be the subset of function satisfying

$$v(\mathbf{y},\theta) = z(\mathbf{y}) - \theta, \tag{7}$$

where $z(\mathbf{y})$ is a function only of $\mathbf{y}$.

Consequently, the MMSEE (6) is the solution of the following constrained optimization problem

$$\begin{cases} \min\limits_{v} \int_{\Theta}\int_{\Omega} v^2(\mathbf{y},\theta)\, p(\mathbf{y},\theta)\, d\mathbf{y} d\theta \\ \text{subject to } v(\mathbf{y},\theta) \in \mathcal{C}_1 \end{cases} \tag{8}$$

Let $\mathcal{F}$ be the set of functions, $f(\mathbf{y},\theta)$, such that $\forall v(\mathbf{y},\theta) \in \mathcal{L}_p^2$

$$\begin{cases} \lim\limits_{\theta \to \pm\infty} v(\mathbf{y},\theta)\, f(\mathbf{y},\theta) = 0, \\ \lim\limits_{\theta \to \pm\infty} \frac{\partial f(\mathbf{y},\theta)}{\partial \theta} = 0. \end{cases} \tag{9}$$

Let us now introduce the three following subsets of functions belonging to $\mathcal{L}_p^2$

- Subset $\mathcal{C}_2$

$$\mathcal{C}_2 = \left\{ v(\mathbf{y},\theta) \in \mathcal{L}_p^2 \middle/ \forall f \in \mathcal{F},\ \int_{\Theta}\int_{\Omega} v(\mathbf{y},\theta) \frac{\partial f(\mathbf{y},\theta)}{\partial \theta} d\mathbf{y} d\theta = \int_{\Theta}\int_{\Omega} f(\mathbf{y},\theta)\, d\mathbf{y} d\theta \right\}. \tag{10}$$

- Subset $\mathcal{C}_3$

$$\mathcal{C}_3 = \left\{ v(\mathbf{y},\theta) \in \mathcal{L}_p^2 \middle/ \begin{array}{c} \forall f \in \mathcal{F},\ \text{and} \int_{\Theta}\int_{\Omega} f(\mathbf{y},\theta)\, d\mathbf{y} d\theta = 1,\ \text{and } \forall h \text{ such that } \theta + h \in \Theta, \\ \int_{\Theta}\int_{\Omega} v(\mathbf{y},\theta)\left(f(\mathbf{y},\theta+h) - f(\mathbf{y},\theta)\right) d\mathbf{y} d\theta = h \end{array} \right\}. \tag{11}$$

- Subset $\mathcal{C}_4$

$$\mathcal{C}_4 = \left\{ v(\mathbf{y},\theta) \in \mathcal{L}_p^2 \middle/ \begin{array}{c} \forall f \in \mathcal{F},\ \text{and} \int_{\Theta}\int_{\Omega} f(\mathbf{y},\theta)\, d\mathbf{y} d\theta = 1,\ \text{and } \forall h \text{ such that } \theta \pm h \in \Theta,\ \text{and, } \forall s \in [0,1], \\ \int_{\Theta}\int_{\Omega} v(\mathbf{y},\theta)\left[L^s(\mathbf{y},\theta+h,\theta) - L^{1-s}(\mathbf{y},\theta-h,\theta)\right] f(\mathbf{y},\theta)\, d\mathbf{y} d\theta \\ = h \int_{\Theta}\int_{\Omega} L^{1-s}(\mathbf{y},\theta-h,\theta)\, f(\mathbf{y},\theta)\, d\theta d\mathbf{y} \end{array} \right\}, \tag{12}$$

with $L(\mathbf{y},\eta,\theta) = \frac{f(\mathbf{y},\eta)}{f(\mathbf{y},\theta)}$.

Theorem 1 below shows that, although these four subsets are generated in a different manner, they are the same.

*Theorem 1:*

$$\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3 = \mathcal{C}_4. \tag{13}$$

The proof of theorem 1 (13) is given in Appendix A.

Consequently, the MMSEE (6) (Best Bayesian Bound) is the solution of the following constrained optimization problem $\forall \mathcal{C}_i \;\; i = 1, \ldots, 4$

$$\begin{cases} \min\limits_{v} \int\limits_{\Theta}\int\limits_{\Omega} v^2(\mathbf{y}, \theta)\, p(\mathbf{y}, \theta)\, d\mathbf{y} d\theta \\ \text{subject to } v(\mathbf{y}, \theta) \in \mathcal{C}_i \end{cases} \tag{14}$$

## III. Weiss-Weinstein family unification

In the light of the previous analysis, it appears a natural manner to introduce Bayesian bound lower than the MMSEE. Indeed, if $\mathcal{P}_i$ is a subset of $\mathcal{C}_i$, the solution of

$$\begin{cases} \min\limits_{v} \int\limits_{\Theta}\int\limits_{\Omega} v^2(\mathbf{y}, \theta)\, p(\mathbf{y}, \theta)\, d\mathbf{y} d\theta \\ \text{subject to } v(\mathbf{y}, \theta) \in \mathcal{P}_i \end{cases} \tag{15}$$

will be also a lower bound of the MMSEE. In this paper, we will first show that an appropriate choice of $\mathcal{P}_i$ leads to the Bayesian bounds of the Weiss-Weinstein family. Second, we will show how this approach can be used in order to build new minimal bounds, particularly, by solving the following constrained optimization problem

$$\begin{cases} \min\limits_{v} \int\limits_{\Theta}\int\limits_{\Omega} v^2(\mathbf{y}, \theta)\, p(\mathbf{y}, \theta)\, d\mathbf{y} d\theta \\ \text{subject to } v(\mathbf{y}, \theta) \in \mathcal{P}_i \cap \mathcal{P}_j \quad i \neq j \end{cases} \tag{16}$$

In this section, we restrict $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ in order to obtain a general framework to create minimal bounds. Then, by way of a constrained optimization problem for which we give an explicit solution we unify the bounds of the Weiss-Weinstein family.

### A. *A general class of lower bounds based on $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$*

Thanks to Theorem 1, we have proposed four equivalent sets of functions $v(\mathbf{y}, \theta)$ leading to the MMSEE. Note that this equivalence holds for

$$\forall f(\mathbf{y}, \theta) \in \mathcal{F} \text{ in the subset } \mathcal{C}_2, \tag{17}$$

$$\begin{cases} \forall f(\mathbf{y}, \theta) \in \mathcal{F} \text{ such that } \int\limits_{\Theta}\int\limits_{\Omega} f(\mathbf{y}, \theta)\, d\mathbf{y} d\theta = 1, \\ \forall h \text{ such that } \theta + h \in \Theta. \end{cases} \quad \text{in the subset } \mathcal{C}_3, \tag{18}$$

$$\begin{cases} \forall f(\mathbf{y}, \theta) \in \mathcal{F} \text{ such that } \int\limits_{\Theta}\int\limits_{\Omega} f(\mathbf{y}, \theta)\, d\mathbf{y} d\theta = 1, \\ \forall h \text{ such that } \theta \pm h \in \Theta, \\ \forall s \in [0, 1]. \end{cases} \quad \text{in the subset } \mathcal{C}_4. \tag{19}$$

Consequently, if we take a finite set of functions $f(\mathbf{y}, \theta)$, a finite set of values $h$, and a finite set of values $s$, we will find bounds lower than the best Bayesian bounds and consequently a general class of minimal bounds on the MSE.

In this way, $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ are restricted, respectively, as follows

$\mathcal{P}_2$**:**

$$\left\{\begin{array}{l}\text{for a finite set of functions } f_i\left(\mathbf{y},\theta\right)\in\mathcal{F},\quad i=1...r,\\ \int\limits_{\Theta}\int\limits_{\Omega}v\left(\mathbf{y},\theta\right)\frac{\partial f_i(\mathbf{y},\theta)}{\partial\theta}d\mathbf{y}d\theta{=}\int\limits_{\Theta}\int\limits_{\Omega}f_i\left(\mathbf{y},\theta\right)d\mathbf{y}d\theta.\end{array}\right. \tag{20}$$

$\mathcal{P}_3$**:**

$$\left\{\begin{array}{l}\text{for a particular function } f\left(\mathbf{y},\theta\right)\in\mathcal{F}\text{ such that }\int\limits_{\Theta}\int\limits_{\Omega}f\left(\mathbf{y},\theta\right)d\mathbf{y}d\theta=1,\\ \text{for a finite set of value } h_i\text{ such that }\theta+h_i\in\Theta,\quad i=1...r,\\ \int\limits_{\Theta}\int\limits_{\Omega}v\left(\mathbf{y},\theta\right)\left(f\left(\mathbf{y},\theta+h_i\right)-f\left(\mathbf{y},\theta\right)\right)d\mathbf{y}d\theta{=}h_i.\end{array}\right. \tag{21}$$

$\mathcal{P}_4$**:**

$$\left\{\begin{array}{l}\text{for a particular function } f\left(\mathbf{y},\theta\right)\in\mathcal{F}\text{ such that }\int\limits_{\Theta}\int\limits_{\Omega}f\left(\mathbf{y},\theta\right)d\mathbf{y}d\theta=1,\\ \text{for a finite set of value } h_i\text{ such that }\theta+h_i\in\Theta,\quad i=1...r,\\ \text{for a finite set of value } s_i\text{ such that }s_i\in\left[0,1\right],\quad i=1...r,\\ \int\limits_{\Theta}\int\limits_{\Omega}v\left(\mathbf{y},\theta\right)\left[L^{s_j}\left(\mathbf{y},\theta+h_i,\theta\right)-L^{1-s_j}\left(\mathbf{y},\theta-h_i,\theta\right)\right]f\left(\mathbf{y},\theta\right)d\mathbf{y}d\theta=h_i\int\limits_{\Theta}\int\limits_{\Omega}L^{1-s_j}\left(\mathbf{y},\theta-h_i,\theta\right)f\left(\mathbf{y},\theta\right)d\theta d\mathbf{y},\end{array}\right. \tag{22}$$

with $L\left(\mathbf{y},\eta,\theta\right)=\frac{f(\mathbf{y},\eta)}{f(\mathbf{y},\theta)}$.

$\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$ define a set of finite constraints, and the problem (15) becomes a classical linear constrained optimization problem

$$\left\{\begin{array}{l}\min\limits_{v}\int\limits_{\Theta}\int\limits_{\Omega}v^2\left(\mathbf{y},\theta\right)p\left(\mathbf{y},\theta\right)d\mathbf{y}d\theta\\ \text{subject to }\int\limits_{\Theta}\int\limits_{\Omega}v\left(\mathbf{y},\theta\right)\tilde{g}_k\left(\mathbf{y},\theta\right)d\mathbf{y}d\theta=c_k\quad k=1...K,\end{array}\right. \tag{23}$$

where $g_k\left(\mathbf{y},\theta\right)$ and $c_k$ are the functions and the scalars involved in $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$.

For $\mathcal{P}_2$

$$\tilde{g}_k\left(\mathbf{y},\theta\right)=\frac{\partial f_k\left(\mathbf{y},\theta\right)}{\partial\theta},\ c_k=\int\limits_{\Theta}\int\limits_{\Omega}f_k\left(\mathbf{y},\theta\right)d\mathbf{y}d\theta\text{ and }K=r. \tag{24}$$

For $\mathcal{P}_3$

$$\tilde{g}_k\left(\mathbf{y},\theta\right)=f\left(\mathbf{y},\theta+h_k\right)-f\left(\mathbf{y},\theta\right),\ c_k=h_k\text{ and }K=r. \tag{25}$$

For $\mathcal{P}_4$

$$\left\{\begin{array}{l}\tilde{g}_k\left(\mathbf{y},\theta\right)=\left[L^{s_k}\left(\mathbf{y},\theta+h_k,\theta\right)-L^{1-s_k}\left(\mathbf{y},\theta-h_k,\theta\right)\right]f\left(\mathbf{y},\theta\right),\\ c_k=h_k\int\limits_{\Theta}\int\limits_{\Omega}L^{1-s_k}\left(\mathbf{y},\theta-h_k,\theta\right)f\left(\mathbf{y},\theta\right)d\theta d\mathbf{y},\\ \text{and }K=r.\end{array}\right. \tag{26}$$

Theorem 2 below gives the solution of the problem (23). Note that this theorem has already been used in the case of a deterministic parameter in [17].

*Theorem 2:* Let $\mathbf{x}\in\mathbb{R}^N$ be a real vector and $p\left(\mathbf{x}\right)$ and $q\left(\mathbf{x}\right)$ be two functions of $\mathbb{R}^N\rightarrow\mathbb{R}$. Let

$$\left\langle p\left(\mathbf{x}\right),q\left(\mathbf{x}\right)\right\rangle=\int\limits_{\mathbb{R}^N}p\left(\mathbf{x}\right)q\left(\mathbf{x}\right)d\mathbf{x}, \tag{27}$$

be an inner product of these two functions and its associate norm $\|p(\mathbf{x})\|^2 = \langle p(\mathbf{x}), p(\mathbf{x})\rangle$. Let $u(\mathbf{x})$ and $g_0(\mathbf{x})$,..., $g_K(\mathbf{x})$ be a set of functions of $\mathbb{R}^N \to \mathbb{R}$, and let $c_0$, $c_1$,..., $c_K$ and $K+1$ be real numbers. Then, the solution of the constrained optimization problem leading to the minimum of $\|u(\mathbf{x})\|^2$ under the following $K+1$ constraints

$$\langle u(\mathbf{x}), g_k(\mathbf{x})\rangle = c_k \quad k = 0, ..., K, \tag{28}$$

is given by

$$\begin{cases} \min_u \|u(\mathbf{x})\|^2 = \mathbf{c}^T \mathbf{G}^{-1} \mathbf{c}, \\ \text{subject to (28)}, \end{cases} \tag{29}$$

with

$$\mathbf{c} = \begin{bmatrix} c_0 & c_1 & \cdots & c_K \end{bmatrix}^T, \tag{30}$$

and

$$\{\mathbf{G}\}_{m,n} = \langle g_m(\mathbf{x}), g_n(\mathbf{x})\rangle . \tag{31}$$

The proof of Theorem 2 (29) is given in Appendix B.

### *B. Application to the Weiss-Weinstein family*

Using (29), $\mathcal{P}_2$, $\mathcal{P}_3$, and $\mathcal{P}_4$, we have built a general framework to obtain Bayesian minimal bounds on the MSE. In this section, we apply this framework and we revisit the Bayesian bounds of the Weiss-Weinstein family. Let $\mathbf{x} = \begin{bmatrix} \mathbf{y}^T & \theta \end{bmatrix}$ and $u(\mathbf{x}) = v(\mathbf{y}, \theta)\sqrt{p(\mathbf{y}, \theta)}$ (*i.e.* $\tilde{g}_k(\mathbf{y}, \theta) = \sqrt{p(\mathbf{y}, \theta)} g_k(\mathbf{y}, \theta)$). Note that Theorem 2 still holds for a set of complex observations $\bar{\mathbf{y}}$ by letting $\mathbf{y} = \begin{bmatrix} \mathrm{Re}\{\bar{\mathbf{y}}^T\} & \mathrm{Im}\{\bar{\mathbf{y}}^T\} \end{bmatrix}^T$.

Moreover, due to the restriction at some particular values of $f(\mathbf{y}, \theta)$, $h$, and $s$, it is still possible to add constraints with our *prior* on the MMSEE in order to achieve tighter bounds. Here we will use the natural constraints of a null bias in terms of the joint probability function; *i.e.*, $\int\limits_{\Theta}\int\limits_{\Omega} v(\mathbf{y}, \theta)\, p(\mathbf{y}, \theta)\, d\mathbf{y} d\theta$=0, where $p(\mathbf{y}, \theta)$ is the joint density of the problem (*i.e.*, $g_0(\mathbf{y},\theta) = \sqrt{p(\mathbf{y},\theta)}$ and $c_0 = 0$).

*a) Bayesian Cramér-Rao bound:* By using the set $\mathcal{P}_2$ with $K = 1$ and $f_1(\mathbf{y}, \theta) = p(\mathbf{y}, \theta)$ (consequently, $\int\limits_{\Theta}\int\limits_{\Omega} f_1(\mathbf{y}, \theta)\, d\mathbf{y} d\theta$=1), we obtain the following set of constraints:

$$\begin{cases} \mathbf{c} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \\ g_0(\mathbf{y},\theta) = \sqrt{p(\mathbf{y},\theta)}, \\ g_1(\mathbf{y},\theta) = \frac{1}{\sqrt{p(\mathbf{y},\theta)}} \frac{\partial p(\mathbf{y},\theta)}{\partial \theta}. \end{cases} \tag{32}$$

Matrix $\mathbf{G}$ involved in Theorem 2 is

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & \int\limits_{\Theta}\int\limits_{\Omega} \left(\frac{\partial \ln p(\mathbf{y},\theta)}{\partial \theta}\right)^2 p(\mathbf{y},\theta)\, d\mathbf{y} d\theta \end{pmatrix}, \tag{33}$$

since $\int\limits_{\Theta}\int\limits_{\Omega} \frac{\partial p(\mathbf{y},\theta)}{\partial \theta} d\mathbf{y} d\theta = 0$.

Finally

$$\mathbf{c}^T\mathbf{G}^{-1}\mathbf{c} = \left(\int\limits_{\Theta}\int\limits_{\Omega}\left(\frac{\partial \ln p(\mathbf{y},\theta)}{\partial\theta}\right)^2 p(\mathbf{y},\theta)\, d\mathbf{y}d\theta\right)^{-1} = BCRB, \tag{34}$$

which is the Bayesian Cramér-Rao bound [20] page 72, and 84.

*b) Bayesian Bhattacharyya bound:* By using the set $\mathcal{P}_2$ with $K = r$ and $f_k(\mathbf{y},\theta) = \frac{\partial^{k-1}p(\mathbf{y},\theta)}{\partial\theta^{k-1}}$, we obtain the following set of constraints:

$$\begin{cases} \mathbf{c} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T, \\ g_0(\mathbf{y},\theta) = \sqrt{p(\mathbf{y},\theta)}, \\ g_k(\mathbf{y},\theta) = \frac{1}{\sqrt{p(\mathbf{y},\theta)}}\frac{\partial^k p(\mathbf{y},\theta)}{\partial\theta^k} \quad k = 1,\ldots,K. \end{cases} \tag{35}$$

We assume that the joint probability density function is such that $\lim\limits_{\theta\to\pm\infty}\frac{\partial^{k-1}p(\mathbf{y},\theta)}{\partial\theta^{k-1}} = 0$ for $k = 3,\ldots,K$. With this assumption and Eqn. (9), we have ,

$$\mathbf{c}^T\mathbf{G}^{-1}\mathbf{c} = \left\{\mathbf{B}^{-1}\right\}_{1,1} = BhattB, \tag{36}$$

where

$$\{\mathbf{B}\}_{i,j} = \int\limits_{\Theta}\int\limits_{\Omega}\frac{1}{p(\mathbf{y},\theta)}\frac{\partial^i p(\mathbf{y},\theta)}{\partial\theta^i}\frac{\partial^j p(\mathbf{y},\theta)}{\partial\theta^j}d\mathbf{y}d\theta, \tag{37}$$

which is the Bayesian Bhattacharyya bound [20] page 149.

*c) Bobrovsky-MayerWolf-Zakaï bound:* By using the set $\mathcal{P}_2$ with $K = 1$ and $f_1(\mathbf{y},\theta) = q(\mathbf{y},\theta)\,p(\mathbf{y},\theta)$, where $q(\mathbf{y},\theta)$ is any function such that $f_1(\mathbf{y},\theta)$ satisfies (9), we obtain the following set of constraints:

$$\begin{cases} \mathbf{c} = \begin{bmatrix} 0 & \int\limits_{\Theta}\int\limits_{\Omega} q(\mathbf{y},\theta)\,p(\mathbf{y},\theta)\,d\mathbf{y}d\theta \end{bmatrix}^T, \\ g_0(\mathbf{y},\theta) = \sqrt{p(\mathbf{y},\theta)}, \\ g_1(\mathbf{y},\theta) = \frac{1}{\sqrt{p(\mathbf{y},\theta)}}\frac{\partial[p(\mathbf{y},\theta)q(\mathbf{y},\theta)]}{\partial\theta}. \end{cases} \tag{38}$$

Due to (9), $\int\limits_{\Theta}\int\limits_{\Omega}\frac{\partial q(\mathbf{y},\theta)p(\mathbf{y},\theta)}{\partial\theta}d\mathbf{y}d\theta = 0$ and the matrix $\mathbf{G}$ involved in Theorem 2 is

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & \int\limits_{\Theta}\int\limits_{\Omega}\frac{1}{\sqrt{p(\mathbf{y},\theta)}}\frac{\partial[p(\mathbf{y},\theta)q(\mathbf{y},\theta)]}{\partial\theta}d\mathbf{y}d\theta \end{pmatrix}. \tag{39}$$

Finally,

$$\mathbf{c}^T\mathbf{G}^{-1}\mathbf{c} = \frac{\left(\int\limits_{\Theta}\int\limits_{\Omega} q(\mathbf{y},\theta)\,p(\mathbf{y},\theta)\,d\theta d\mathbf{y}\right)^2}{\int\limits_{\Theta}\int\limits_{\Omega}\frac{1}{p(\mathbf{y},\theta)}\left(\frac{\partial[p(\mathbf{y},\theta)q(\mathbf{y},\theta)]}{\partial\theta}\right)^2 d\mathbf{y}d\theta} = BMZB(q(\mathbf{y},\theta)). \tag{40}$$

We recognize the Bobrovsky-MayerWolf-Zakaï bound [39], which is an extension of the Bayesian Cramér-Rao bound, since

$$BMZB\left(1\right) = BCRB. \tag{41}$$

*d) Bobrovsky-Zakaï bound:* We choose here that the particular value of $f\left(\mathbf{y},\theta\right) = p\left(\mathbf{y},\theta\right)$, the joint density probability function of the problem. Consequently, $\int\limits_{\Theta}\int\limits_{\Omega} f\left(\mathbf{y},\theta\right) d\mathbf{y}d\theta$=1.

By using the set $\mathcal{P}_3$ with $K = 1$, we obtain the following set of constraints:

$$\begin{cases} \mathbf{c} = \begin{bmatrix} 0 & h \end{bmatrix}^T, \\ g_0\left(\mathbf{y},\theta\right) = \sqrt{p\left(\mathbf{y},\theta\right)}, \\ g_1\left(\mathbf{y},\theta\right) = \frac{p(\mathbf{y},\theta+h)-p(\mathbf{y},\theta)}{\sqrt{p(\mathbf{y},\theta)}}. \end{cases} \tag{42}$$

Matrix $\mathbf{G}$ involved in Theorem 2 is

$$\mathbf{G} = \begin{pmatrix} 1 & 0 \\ 0 & \int\limits_{\Theta}\int\limits_{\Omega} \frac{(p(\mathbf{y},\theta+h)-p(\mathbf{y},\theta))^2}{p(\mathbf{y},\theta)} d\mathbf{y}d\theta \end{pmatrix}. \tag{43}$$

Finally,

$$\begin{aligned} \mathbf{c}^T\mathbf{G}^{-1}\mathbf{c} &= \frac{h^2}{\int\limits_{\Theta}\int\limits_{\Omega} \frac{p^2(\mathbf{y},\boldsymbol{\theta}+h)}{p(\mathbf{y},\boldsymbol{\theta})} d\mathbf{y}d\theta - 1} \\ &= BZB\left(h\right). \end{aligned} \tag{44}$$

Since $h$ is a parameter left to the user, the highest bound that can be obtained with (44) is given by

$$BZB = \sup_h BZB\left(h\right) = \sup_h \frac{h^2}{\int\limits_{\Theta}\int\limits_{\Omega} \frac{p^2(\mathbf{y},\theta+h)}{p(\mathbf{y},\theta)} d\mathbf{y}d\theta - 1}, \tag{45}$$

which is the Bobrovsky-Zakaï bound [40].

*e) Reuven-Messer bound:* We choose here that the particular value of $f\left(\mathbf{y},\theta\right) = p\left(\mathbf{y},\theta\right)$, the joint density probability function of the problem. Consequently, $\int\limits_{\Theta}\int\limits_{\Omega} f\left(\mathbf{y},\theta\right) d\mathbf{y}d\theta$=1.

In order to obtain a bound tighter than the Bobrovsky-Zakaï bound (*i.e.*, $\mathcal{P}_3 \to \mathcal{C}_3$), we use the set $\mathcal{P}_3$ with $K = r$. We then obtain the following set of constraints:

$$\begin{cases} \mathbf{c} = \begin{bmatrix} 0 & \mathbf{h}^T \end{bmatrix}^T, \\ g_0\left(\mathbf{y},\theta\right) = \sqrt{p\left(\mathbf{y},\theta\right)}, \\ g_k\left(\mathbf{y},\theta\right) = \frac{p(\mathbf{y},\theta+h_k)-p(\mathbf{y},\theta)}{\sqrt{p(\mathbf{y},\theta)}} \quad k = 1,\ldots,r. \end{cases} \tag{46}$$

where $\mathbf{h} = \begin{bmatrix} h_1 & \cdots & h_r \end{bmatrix}^T$.

Matrix $\mathbf{G}$ involved in Theorem 2 is

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{D} & \\ 0 & & & \end{pmatrix}, \tag{47}$$

where $\mathbf{D}$ $(r \times r)$ is defined as

$$\begin{aligned}\{\mathbf{D}\}_{i,,j} &= \int\limits_{\Theta}\int\limits_{\Omega} \frac{\left(p\left(\mathbf{y},\theta+h_i\right)-p\left(\mathbf{y},\theta\right)\right)\left(p\left(\mathbf{y},\theta+h_j\right)-p\left(\mathbf{y},\theta\right)\right)}{p\left(\mathbf{y},\theta\right)} d\mathbf{y}d\theta \\ &= \int\limits_{\Theta}\int\limits_{\Omega} \frac{p\left(\mathbf{y},\theta+h_i\right)p\left(\mathbf{y},\theta+h_j\right)}{p\left(\mathbf{y},\theta\right)} d\mathbf{y}d\theta - 1. \end{aligned} \tag{48}$$

Finally,

$$\begin{aligned}\mathbf{c}^T\mathbf{G}^{-1}\mathbf{c} &= \mathbf{h}^T\mathbf{D}^{-1}\mathbf{h} \\ &= RMB\left(\mathbf{h}\right). \end{aligned} \tag{49}$$

As for the Bobrovsky-Zakaï bound, since $\mathbf{h}$ is a parameter vector left to the user, the highest bound that can be obtained with (49) is given by

$$RMB = \sup_{\mathbf{h}} RMB\left(\mathbf{h}\right) = \sup_{\mathbf{h}} \mathbf{h}^T\mathbf{D}^{-1}\mathbf{h}, \tag{50}$$

which is a particular case[1] of the Reuven-Messer bound [41].

*f) Weiss-Weinstein bound:* We choose here that the particular value of $f\left(\mathbf{y},\theta\right) = p\left(\mathbf{y},\theta\right)$, the joint density probability function of the problem. Consequently, $\int\limits_{\Theta}\int\limits_{\Omega} f\left(\mathbf{y},\theta\right)d\mathbf{y}d\theta{=}1$.

By using the set $\mathcal{P}_4$ with $K = r$, we obtain the following set of constraints:

$$\begin{cases} \mathbf{c} = \begin{bmatrix} 0 & h_1\mathbb{E}_{\mathbf{y},\theta}\left[L^{1-s_1}\left(\mathbf{y},\theta-h_1,\theta\right)\right] & \cdots & h_k\mathbb{E}_{\mathbf{y},\theta}\left[L^{1-s_k}\left(\mathbf{y},\theta-h_k,\theta\right)\right] \end{bmatrix}^T, \\ g_0\left(\mathbf{y},\theta\right) = \sqrt{p\left(\mathbf{y},\theta\right)}, \\ g_k\left(\mathbf{y},\theta\right) = \sqrt{p\left(\mathbf{y},\theta\right)}\left(L^{s_k}\left(\mathbf{y},\theta+h_k,\theta\right) - L^{1-s_k}\left(\mathbf{y},\theta-h_k,\theta\right)\right) \quad k = 1,\ldots,r. \end{cases} \tag{51}$$

Let

$$\boldsymbol{\xi} = \begin{bmatrix} h_1\mathbb{E}_{\mathbf{y},\theta}\left[L^{1-s_1}\left(\mathbf{y},\theta-h_1,\theta\right)\right] & \cdots & h_r\mathbb{E}_{\mathbf{y},\theta}\left[L^{1-s_r}\left(\mathbf{y},\theta-h_r,\theta\right)\right] \end{bmatrix}^T, \tag{52}$$

$$\mathbf{h} = \begin{bmatrix} h_1 & \cdots & h_r \end{bmatrix}^T, \tag{53}$$

$$\mathbf{s} = \begin{bmatrix} s_1 & \cdots & s_r \end{bmatrix}^T. \tag{54}$$

The application of Theorem 2 leads to

$$\begin{aligned}\mathbf{c}^T\mathbf{G}^{-1}\mathbf{c} &= \boldsymbol{\xi}^T\mathbf{W}^{-1}\boldsymbol{\xi} \\ &= WWB\left(\mathbf{h},\mathbf{s}\right), \end{aligned} \tag{55}$$

where

$$\{\mathbf{W}\}_{i,,j} = \mathbb{E}_{\mathbf{y},\theta}[\left(L^{s_i}\left(\mathbf{y},\theta+h_i,\theta\right) - L^{1-s_i}\left(\mathbf{y},\theta-h_i,\theta\right)\right)\left(L^{s_j}\left(\mathbf{y},\theta+h_j,\theta\right) - L^{1-s_j}\left(\mathbf{y},\theta-h_j,\theta\right)\right)]. \tag{56}$$

[1]In 1997, Reuven and Messer proposed a hybrid minimal bound based on the Barankin bound for both random and non-random vector of parameters. Here, only the random case is considered.

As for the Bobrovsky-Zakaï bound and the Reuven-Messer bound, since $\mathbf{h}$ and $\mathbf{s}$ are parameter vectors left to the user, the highest bound that can be obtained with (55) is given by

$$WWB = \sup_{\mathbf{h},\mathbf{s}} WWB\left(\mathbf{h},\mathbf{s}\right) = \sup_{\mathbf{h},\mathbf{s}} \boldsymbol{\xi}^T \mathbf{W}^{-1} \boldsymbol{\xi}. \tag{57}$$

We recognize the Weiss-Weinstein bound [42].

## IV. New minimal bounds

The framework proposed in the last section allows us to rederive all the bounds of the Weiss-Weinstein family by way of a constrained optimization problem. But this framework is also useful for deriving new lower bounds. In this section, we propose two lower bounds.

### A. Some global classes of Bhattacharyya bounds

In [39], Bobrovsky, Mayer-Wolf, and Zakaï propose an extension of the Bayesian Cramér-Rao bound given by Equation (40). The advantage of this bound is the degree of freedom given by $q\left(\mathbf{y},\theta\right)$. Indeed, the authors give some examples for which use of a properly chosen function $q\left(\mathbf{y},\theta\right)$ leads to useful bounds. Moreover, when $p\left(\mathbf{y},\theta\right)$ does not satisfy the regularity assumption given in [20] (*e.g.*, for uniform random variables), a properly chosen $q\left(\mathbf{y},\theta\right)$ can solve the problem. Here we obtain an extension of this bound and of the Bayesian Bhattacharyya bound in a straightforward manner by mixing the constraints of the Bobrovsky-MayerWolf-Zakaï bound and the constraints of the Bayesian Bhattacharyya bound.

By using the set $\mathcal{P}_2$ with $K = r$ and $f_k\left(\mathbf{y},\theta\right) = \frac{\partial^{k-1}\left[q(\mathbf{y},\theta)p(\mathbf{y},\theta)\right]}{\partial\theta^{k-1}}$, where $q\left(\mathbf{y},\theta\right)$ is any function such that $f_k\left(\mathbf{y},\theta\right)$ satisfies (9), we obtain the following set of constraints:

$$\begin{cases} \mathbf{c} = \begin{bmatrix} 0 & \int\limits_{\Theta}\int\limits_{\Omega} q\left(\mathbf{y},\theta\right) p\left(\mathbf{y},\theta\right) d\mathbf{y}d\theta & 0 & \cdots & 0 \end{bmatrix}^T, \\ g_0\left(\mathbf{y},\theta\right) = \sqrt{p\left(\mathbf{y},\theta\right)}, \\ g_k\left(\mathbf{y},\theta\right) = \frac{1}{\sqrt{p(\mathbf{y},\theta)}} \frac{\partial^k\left[q(\mathbf{y},\theta)p(\mathbf{y},\theta)\right]}{\partial\theta^k} \quad k = 1,\ldots,K. \end{cases} \tag{58}$$

We assume that the functions $q\left(\mathbf{y},\theta\right)$ and $p\left(\mathbf{y},\theta\right)$ are such that $\lim\limits_{\theta\to\pm\infty} \frac{\partial^{k-1}q(\mathbf{y},\theta)p(\mathbf{y},\theta)}{\partial\theta^{k-1}} = 0 \quad$ for $k = 3,\ldots,K$. With this assumption and Eqn. (9), we have ,

$$\mathbf{c}^T\mathbf{G}^{-1}\mathbf{c} = \frac{\left(\int\limits_{\Theta}\int\limits_{\Omega} q\left(\mathbf{y},\theta\right) p\left(\mathbf{y},\theta\right) d\mathbf{y}d\theta\right)^2}{\left\{\bar{\mathbf{B}}^{-1}\right\}_{1,1}} \tag{59}$$

where

$$\left\{\bar{\mathbf{B}}\right\}_{i,j} = \int\limits_{\Theta}\int\limits_{\Omega} \frac{1}{p\left(\mathbf{y},\theta\right)} \frac{\partial^i\left[q\left(\mathbf{y},\theta\right)p\left(\mathbf{y},\theta\right)\right]}{\partial\theta^i} \frac{\partial^j\left[q\left(\mathbf{y},\theta\right)p\left(\mathbf{y},\theta\right)\right]}{\partial\theta^j} d\mathbf{y}d\theta. \tag{60}$$

*B. The Bayesian Abel bound*

In this section, we propose a new minimal bound on the MSE based on our framework and on the Abel works on deterministic bounds [29], [30]. In the deterministic parameter context, the Cramér-Rao bound and the Bhattacharyya bound account for the *small estimation error* (near the true value of the parameters). The Chapman-Robbins bound and the Barankin bound account for the *large estimation error* generally due to the appearance of outliers which creates the performance breakdown phenomena. In [29] [30], Abel combined the two kinds of bounds in order to obtain a bound that accounts for both local and large errors. The obtained deterministic Abel bound leads to a generalization of the Cramér-Rao, the Bhattacharyya, the Chapman-Robbins, and the Barankin bounds. As the deterministic bounds, the Bayesian Cramér-Rao bound and the Bayesian Bhattacharyya bound are *small error* bounds, as compared to the Bobrovsky-Zakaï bound and the Reuven-Messer bound which are *large error* bounds. The purpose here is to apply the idea of Abel in the Bayesian context, i.e. to derive a bound that combines the Bayesian small and large error bounds. This application will be accomplished by way of the constrained optimization problem introduced in the last section. Our Bayesian version of the Abel bound is derived by mixing the constraints of the Reuven-Messer bound and the Bayesian Bhattacharyya bound and, thus, represents a generalization of these bounds. Consequently, we are solving the following constrained optimization problem

$$\begin{cases} \min\limits_{v} \int\limits_{\Theta}\int\limits_{\Omega} v^2\left(\mathbf{y},\theta\right) p\left(\mathbf{y},\theta\right) d\mathbf{y} d\theta \\ \text{subject to } v\left(\mathbf{y},\theta\right) \in \mathcal{P}_2 \cap \mathcal{P}_3 \end{cases} \tag{61}$$

By combining the Bayesian Bhattacharyya constraints (35) and the Reuven-Messer constraints (46), i.e, by concatenating both vectors $\mathbf{g} = \left[g_0\left(\mathbf{y},\theta\right), g_1\left(\mathbf{y},\theta\right), ..., g_K\left(\mathbf{y},\theta\right)\right]^T$ and $\mathbf{c}$ from the Bayesian Bhattacharyya bound of order $m$ and from the Reuven-Messer bound of order $r$, we obtain the following new set of $K = m + r + 1$ constraints[2],

$$\mathbf{g} = \frac{1}{\sqrt{p\left(\mathbf{y},\theta\right)}} \begin{bmatrix} p\left(\mathbf{y},\theta\right) \\ \frac{\partial p\left(\mathbf{y},\theta\right)}{\partial\theta} \\ \vdots \\ \frac{\partial^m p\left(\mathbf{y},\theta\right)}{\partial\theta^m} \\ ------ \\ p\left(\mathbf{y},\theta+h_1\right) - p\left(\mathbf{y},\theta\right) \\ \vdots \\ p\left(\mathbf{y},\theta+h_r\right) - p\left(\mathbf{y},\theta\right) \end{bmatrix} \text{ and } \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -- \\ h_1 \\ \vdots \\ h_r \end{bmatrix}. \tag{62}$$

The calculus are detailed in Appendix C, and the theorem 2 leads to

$$\mathbf{c}^T\mathbf{G}^{-1}\mathbf{c} = BAB_{m,r}\left(\mathbf{h}\right) = \boldsymbol{\alpha}^T\mathbf{B}^{-1}\boldsymbol{\alpha} + \mathbf{u}^T\mathbf{J}^{-1}\mathbf{u}, \tag{63}$$

[2]The first constraint of the two bounds is the same.

with

$$\begin{cases} \mathbf{u} = \mathbf{\Gamma}\mathbf{B}^{-1}\boldsymbol{\alpha} - \mathbf{h},\ r \times 1, \\ \mathbf{J} = \mathbf{D} - \mathbf{\Gamma}\mathbf{B}^{-1}\mathbf{\Gamma}^{T},\ r \times r, \\ \boldsymbol{\alpha} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{T},\ m \times 1, \\ \mathbf{h} = \begin{bmatrix} h_1 & h_2 & \cdots & h_r \end{bmatrix}^{T},\ r \times 1, \\ \{\mathbf{D}\}_{i,,j} = \int\limits_{\Theta}\int\limits_{\Omega} \frac{p(\mathbf{y},\theta+h_i)p(\mathbf{y},\theta+h_j)}{p(\mathbf{y},\theta)} d\mathbf{y}d\theta - 1,\ r \times r, \\ \{\mathbf{B}\}_{i,j} = \int\limits_{\Theta}\int\limits_{\Omega} \frac{1}{p(\mathbf{y},\theta)} \frac{\partial^i p(\mathbf{y},\theta)}{\partial\theta^i} \frac{\partial^j p(\mathbf{y},\theta)}{\partial\theta^j} d\mathbf{y}d\theta,\ m \times m, \\ \{\mathbf{\Gamma}\}_{i,j} = \int\limits_{\Theta}\int\limits_{\Omega} \frac{p(\mathbf{y},\theta+h_i)}{p(\mathbf{y},\theta)} \frac{\partial^j p(\mathbf{y},\theta)}{\partial\theta^j} d\mathbf{y}d\theta,\ r \times m. \end{cases} \tag{64}$$

Let us note that the first term on right hand side of (63) is equal to $BAB_{m,0}$, which is the Bayesian Bhattacharyya bound of order $m$, and that $BAB_{0,r}(\mathbf{h})$ is the Reuven-Messer bound of order $r$. We have previously shown that problem (8) leads to the MMSEE (the best Bayesian bound). Here, from the increase of constraints, it follows that the Bayesian Abel bound is (for $r$ and $m$ fixed) a better approximation of the best Bayesian bound than the Bayesian Bhattacharyya bound of order $m$ and the Reuven-Messer bound of order $r$.

The Bayesian Abel bound as the Reuven-Messer bound depends on $r$ free parameters $h_1, ..., h_r$. Then, a maximization over these parameters is desired to obtain the highest bound. Therefore, the best Bayesian Abel bound is given by

$$BAB_{m,r} = \sup_{\mathbf{h}_r} \left( \boldsymbol{\alpha}^T \mathbf{B}^{-1} \boldsymbol{\alpha} + \mathbf{u}^T \mathbf{J}^{-1} \mathbf{u} \right). \tag{65}$$

This multidimensional optimization brings with it a huge computational cost. A possible alternative is given by noting that the Bayesian Cramér-Rao bound is a particular case of the Bayesian Bhattacharyya bound (single derivative) and that the Bobrovsky-Zakaï bound is a particular case of the Reuven-Messer bound (single test point). Therefore, finding a tractable form of the Bayesian Abel bound in the case where $m = 1$ and $r = 1$ could be interesting, since the obtained bound will be tighter than both the Bayesian Cramér-Rao bound and the Bobrovsky-Zakaï bound with a low computational cost. In this case, Equation (65) becomes straightforwardly

$$BAB_{1,1} = \sup_{h} \frac{BCRB^{-1} + BZB^{-1}(h) - 2\phi(h)}{BCRB^{-1} BZB^{-1}(h) - \phi^2(h)}, \tag{66}$$

where *BCRB* is the Bayesian Cramér-Rao bound, *BZB* is the Bobrovsky-Zakaï bound, and

$$\phi(h) = \frac{1}{h} \int\limits_{\Theta} \int\limits_{\Omega} \frac{\partial \ln p(\mathbf{y}, \theta)}{\partial \theta} p(\mathbf{y}, \theta + h) \, d\mathbf{y} d\theta. \tag{67}$$

Equation (66) is interesting, since if the Bayesian Cramér-Rao bound and the Bobrovsky-Zakaï bound are available for a given problem, the evaluation of the $BAB_{1,1}$ requires only the computation of $\phi(h)$.

## V. Bayesian bounds for Signal Processing problems

In this section, we illustrate our previous analysis through a spectral analysis problem. First, we propose several closed-form expressions for the different bounds of the Weiss-Weinstein family (including the proposed Bayesian Abel bound) for a general Gaussian observation model with parameterized mean widely used in the signal processing literature (see, e.g., [68] page 35). Then, we apply these results to the spectral analysis problem. Finally, we give simulation results that compare the different bounds and show the superiority of the Weiss-Weinstein bound.

### A. Gaussian observation model with parameterized mean

We consider the following general observation model:

$$\mathbf{y} = \mathbf{m}(\theta) + \mathbf{n}, \tag{68}$$

where $\mathbf{y}$ is the complex observation vector $(N \times 1)$, $\theta$ is a real unknown parameter, $\mathbf{m}$ is a complex deterministic vector $(N \times 1)$ depending (non-linearly) on $\theta$, and $\mathbf{n}$ is the complex vector $(N \times 1)$ of the noise. The noise is assumed to be circular, Gaussian, with zero mean and with covariance matrix $\sigma^2 \mathbf{I}_N$. The parameter of interest $\theta$ is assumed to have a Gaussian *a priori* probability density function with mean $\mu$ and variance $\sigma_\theta^2$:

$$p(\theta) = \frac{1}{\sqrt{2\pi}\sigma_\theta} e^{-\frac{1}{2\sigma_\theta^2}(\theta-\mu)^2} \tag{69}$$

For this model, the likelihood of the observations is given by

$$p(\mathbf{y}|\,\theta) = \frac{1}{(\pi\sigma^2)^N} e^{-\frac{1}{\sigma^2}(\mathbf{y}-\mathbf{m}(\theta))^H(\mathbf{y}-\mathbf{m}(\theta))}. \tag{70}$$

To the best of our knowledge, only the Cramér-Rao bound expression is known in this case (see [68]).

The Bayesian Bhattacharyya bound requires the calculation of several derivatives of the joint probability function in order to be significantly tighter than the Cramér-Rao bound, which is generally difficult (see [69], Chapter 4, for an example for which the Bhattacharyya bound of order 2 requires much algebraic effort to finally be equal to the Cramér-Rao bound). Consequently, we will not use this bound here.

The details are given in Appendix D.

*1) Bayesian Cramér-Rao bound:*

$$BCRB = \frac{\sigma_\theta^2}{\frac{2\sigma_\theta^2}{\sigma^2}\mathbb{E}_\theta\left[\left\|\frac{\partial \mathbf{m}(\theta)}{\partial\theta}\right\|^2\right] + 1}. \tag{71}$$

*2) Bobrovsky-Zakaï bound:*

$$BZB = \sup_h \frac{h^2}{\int\limits_\Theta \frac{p^2(\theta+h)}{p(\theta)} e^{\frac{2}{\sigma^2}\|\mathbf{m}(\theta+h)-\mathbf{m}(\theta)\|^2} d\theta - 1}. \tag{72}$$

*3) Bayesian Abel bound:* $BAB_{1,1}$ is given by (66):

$$BAB_{1,1} = \sup_h \frac{BCRB^{-1} + BZB^{-1}(h) - 2\phi(h)}{BCRB^{-1}BZB^{-1}(h) - \phi^2(h)}, \tag{73}$$

where

$$\begin{cases} BCRB = \frac{\sigma_\theta^2}{\frac{2\sigma_\theta^2}{\sigma^2}\mathbb{E}_\theta\left[\left\|\frac{\partial \mathbf{m}(\theta)}{\partial\theta}\right\|^2\right]+1}, \\ BZB(h) = \frac{h^2}{\int\limits_\Theta \frac{p^2(\theta+h)}{p(\theta)} e^{\frac{2}{\sigma^2}\|\mathbf{m}(\theta+h)-\mathbf{m}(\theta)\|^2} d\theta - 1}, \end{cases} \tag{74}$$

and

$$\phi(h) = \frac{1}{\sigma_\theta^2} + \frac{2}{h\sigma^2}\mathbb{E}_{\theta+h}\left[\mathrm{Re}\left\{\frac{\partial \mathbf{m}^H(\theta)}{\partial\theta}(\mathbf{m}(\theta+h) - \mathbf{m}(\theta))\right\}\right]. \tag{75}$$

*4) Weiss-Weinstein bound:* We now consider the Weiss-Weinstein bound with one test point, which can be simplified as follows (see [42], Equation (6)):

$$WWB = \sup_{h,s} \frac{h^2 e^{2\eta(s,h)}}{e^{\eta(2s,h)} + e^{\eta(2-2s,-h)} - 2e^{\eta(s,2h)}}, \tag{76}$$

where the key point to evaluate this bound is $\eta(\alpha, \beta)$, which is the semi-invariant moment generating function [70], defined as follows:

$$\eta(\alpha, \beta) = \ln \int_{\Theta} \int_{\Omega} \frac{p^{\alpha}(\mathbf{y}, \theta + \beta)}{p^{\alpha-1}(\mathbf{y}, \theta)} d\mathbf{y} d\theta. \tag{77}$$

This function is given by

$$\eta(\alpha, \beta) = \ln \frac{1}{\sqrt{2\pi}\sigma_\theta} \int_{\Theta} e^{\frac{\alpha(\alpha-1)}{\sigma^2}\|\mathbf{m}(\theta+\beta)-\mathbf{m}(\theta)\|^2 - \frac{1}{2\sigma_\theta^2}\left(\theta-\left(\sqrt{\alpha(\alpha-1)}-\alpha\right)h-\mu\right)\left(\theta+\left(\sqrt{\alpha(\alpha-1)}+\alpha\right)h-\mu\right)} d\theta. \tag{78}$$

### B. Spectral Analysis Problem

We now consider the following observation model involved in spectral analysis:

$$y_k = a e^{j2\pi k\theta} + n_k, \qquad k = 0, \ldots, N-1, \tag{79}$$

where $y_k$ is the $k^{th}$ complex observation. The observations are assumed to be independent. $a$ is the amplitude of the single cisoïde of frequency $\theta$. $\{n_k\}$ is a sequence of random variables assumed complex, circular, i.i.d, Gaussian, with zero mean and variance $\sigma^2$. Consequently the SNR is given by $SNR = \frac{a^2}{\sigma^2}$. The parameter of interest is the frequency $\theta \in \Theta = \left(-\frac{1}{2}, \frac{1}{2}\right]$ which is a Gaussian random variable with mean $\mu$ and variance $\sigma_\theta^2$ (69).

This model is a particular case of the model (68), where

$$\mathbf{m}(\theta) = a\mathbf{s}(\theta), \tag{80}$$

with

$$\mathbf{s}(\theta) = \begin{bmatrix} 1 & e^{j2\pi\theta} & \ldots & e^{j2\pi(N-1)\theta} \end{bmatrix}^T. \tag{81}$$

Let $\mathbf{y} = \begin{bmatrix} y_0 & \cdots & y_{N-1} \end{bmatrix}^T$. The likelihood of the observation is given by

$$p(\mathbf{y}|\theta) = \prod_{k=0}^{N-1} p(y_k|\theta) = \frac{1}{(\pi\sigma^2)^N} e^{-\frac{1}{\sigma^2}\left(\|\mathbf{y}\|^2 - 2a\,\mathrm{Re}\left\{\sum_{k=0}^{N-1} y_k^* e^{j2\pi k\theta}\right\} + Na^2\right)}. \tag{82}$$

Note that, if $\theta$ is assumed to be deterministic and in a digital communications context, some closed-form expressions of deterministic bounds can be found in [56].

The details of the calculus for the Weiss-Weinstein family are given in Appendix E.

*1) Cramér-Rao bound:*

$$BCRB = \frac{\sigma_\theta^2}{SNR\frac{4\pi^2\sigma_\theta^2}{3}N(2N-1)(N-1)+1}. \tag{83}$$

*2) Bobrovsky-Zakaï bound:*

$$BZB = \sup_h \frac{h^2}{e^{4SNR\left(N-\sin^2(\pi hN)-\frac{1}{2}\frac{\sin(2\pi hN)}{\tan(\pi h)}\right)+\frac{h^2}{\sigma_\theta^2}} - 1} \tag{84}$$

*3) Bayesian Abel bound:* The $BAB_{1,1}$ is given by (66)

$$BAB_{1,1} = \sup_{h} \frac{BCRB^{-1} + BZB^{-1}(h) - 2\phi(h)}{BCRB^{-1}BZB^{-1}(h) - \phi^2(h)}, \tag{85}$$

where

$$\begin{cases} BCRB = \frac{1}{SNR} \frac{\sigma_\theta^2}{\frac{4\pi^2\sigma_\theta^2}{3}N(2N-1)(N-1)+1}, \\ BZB(h) = \frac{h^2}{e^{4SNR\left(N-\sin^2(\pi hN)-\frac{1}{2}\frac{\sin(2\pi hN)}{\tan(\pi h)}\right)+\frac{h^2}{\sigma_\theta^2}}-1}, \end{cases} \tag{86}$$

and,

$$\phi(h) = \frac{1}{\sigma_\theta^2} + \frac{2\pi SNR}{h}\left(N\frac{\cos(2\pi hN)}{\tan(\pi h)} - \sin(2\pi hN)\left(\frac{1}{2\sin(\pi h)} + N\right)\right). \tag{87}$$

*4) Weiss-Weinstein bound:* The Weiss-Weinstein bound is given by

$$WWB = \sup_{h,s} \frac{h^2 e^{2\eta(s,h)}}{e^{\eta(2s,h)} + e^{\eta(2-2s,-h)} - 2e^{\eta(s,2h)}}, \tag{88}$$

where $\eta(\alpha,\beta)$ is given by

$$\eta(\alpha,\beta) = \alpha(\alpha-1)\left(2SNR\left(N - \sin^2(\pi\beta N) - \frac{1}{2}\frac{\sin(2\pi\beta N)}{\tan(\pi\beta)}\right) - \frac{\beta^2}{2\sigma_\theta^2}\right). \tag{89}$$

The Weiss-Weinstein bound needs to be optimized over two continuous parameters, which creates significant computational cost. Here, two methods for reducing the computational cost are presented.

- As previously stated, $h$ is chosen on the parameter support which is approximated by $[-3\sigma_\theta, 3\sigma_\theta]$. This support can be reduced to $[0, 3\sigma_\theta]$, since the function is even with respect to $h$. Note that this remark holds for the Bayesian Abel bound and the Bobrovsky-Zakaï bound.
- As proposed by Weiss and Weinstein in [42], it is sometimes a good choice to set $s = 1/2$. This approximation is intuitively justified by the fact that the Weiss-Weinstein bound tends to the Bobrovsky-Zakaï bound when $s$ tends to zero or one. Unfortunately, no sound proof that this result is true in general is available in the literature. If we set $s = 1/2$, $\eta(\alpha,\beta)$ is modified as follows:

$$\begin{cases} \eta\left(\frac{1}{2},h\right) = -\frac{1}{4}\left(2SNR\left(N-\sin^2(\pi hN)-\frac{1}{2}\frac{\sin(2\pi hN)}{\tan(\pi h)}\right)-\frac{h^2}{2\sigma_\theta^2}\right), \\ \eta(1,h) = 0, \\ \eta(1,-h) = 0, \\ \eta\left(\frac{1}{2},2h\right) = -\frac{1}{2}\left(SNR\left(N-\sin^2(2\pi hN)-\frac{1}{2}\frac{\sin(4\pi hN)}{\tan(2\pi h)}\right)-\frac{h^2}{\sigma_\theta^2}\right), \end{cases} \tag{90}$$

  and the modified Weiss-Weinstein bound becomes

$$\overline{WWB} = \sup_{h} \frac{h^2}{2} \frac{e^{-\frac{1}{2}\left(2SNR\left(N-\sin^2(\pi hN)-\frac{1}{2}\frac{\sin(2\pi hN)}{\tan(\pi h)}\right)-\frac{h^2}{2\sigma_\theta^2}\right)}}{1-e^{-\frac{1}{2}\left(SNR\left(N-\sin^2(2\pi hN)-\frac{1}{2}\frac{\sin(4\pi hN)}{\tan(2\pi h)}\right)-\frac{h^2}{\sigma_\theta^2}\right)}}. \tag{91}$$

The resulting bound has approximatively the same computational cost as the BZB and the BAB.

### C. Simulations

In order to illustrate our results on the different bounds, we present here a simulation result for the spectral analysis problem.

We consider a scenario with $N = 15$ observations and, without loss of generality, $a = 1$. The estimator will be the Maximum Likelihood Estimator (MLE) given for this model by

$$\hat{\theta}_{ML} = \arg\min_{\theta} \left[ \|\mathbf{y}\|^2 + Na^2 - 2a \operatorname{Re}\left\{ \sum_{k=0}^{N-1} y_k^* e^{j2\pi k\theta} \right\} \right]. \tag{92}$$

We also use the Maximum A Posteriori (MAP) estimator given by

$$\hat{\theta}_{MAP} = \arg\min_{\theta} \left[ \frac{1}{\sigma^2} \left( \|\mathbf{y}\|^2 + Na^2 - 2a \operatorname{Re}\left\{ \sum_{k=0}^{N-1} y_k^* e^{j2\pi k\theta} \right\} \right) + \frac{\theta^2}{2\sigma_\theta^2} \right]. \tag{93}$$

The global MSE will be computed by using the relation (3) and 1000 Monte-Carlo runs. For the *a priori* probability density function of the parameter of interest, we choose $\mu = 0$ and $\sigma_\theta^2 = \frac{1}{36}$ rad$^2$.

Figure (1) superimposes the global MSE of the MLE and of the MAP estimator, the Cramér-Rao bound, the Bobrovsky-Zakaï bound, the Bayesian Abel bound, and the Weiss-Weinstein bound with optimization over $s$ and $s = 1/2$.

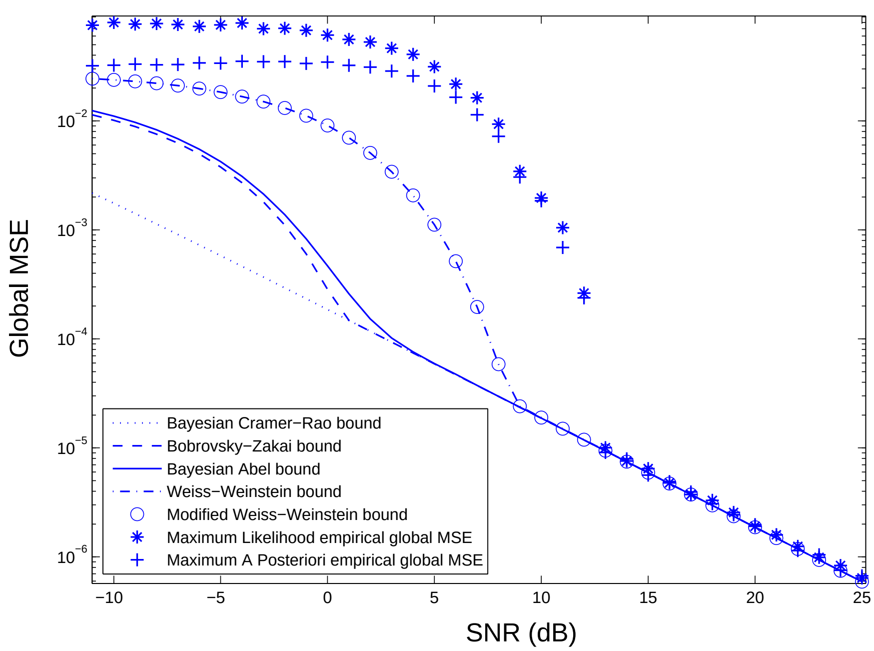


Fig. 1. Comparison of the global MSE of the MLE and of the MAP estimator, the Cramér-Rao bound, the Bobrovsky-Zakaï bound, the Bayesian Abel bound, and the Weiss-Weinstein bound with optimization over $s$ and $s = 1/2$. $N = 15$ observations. $\sigma_\theta^2 = \frac{1}{36}$ rad$^2$.

This figure shows the threshold behavior of both estimators when the SNR decreases. In contrast to the Cramér-Rao bound, the Bobrovsky-Zakaï bound, the Bayesian Abel bound, and the Weiss-Weinstein bound exhibit the threshold phenomena. The Bayesian Abel bound is slightly higher than the Bobrovsky-Zakaï bound and, consequently, leads to a better prediction of the threshold effect with the same computational cost. The Weiss-Weinstein bounds obtained by numerical evaluation of Equations (88) and (91) are the same; therefore, s=1/2 seems to be the optimum value in this problem. As expected by the addition of constraints, the Weiss-Weinstein bounds provide a better prediction of the global MSE of the estimators in comparison with the Bobrovsky-Zakaï bound and the Bayesian Abel bound. The Weiss-Weinstein bound threshold value provides a better approximation of the effective SNR at which the estimators experience the threshold behavior.

## VI. Conclusion

In this paper, we proposed a framework to study the Bayesian minimal bounds on the mean square error of the Weiss-Weinstein family. This framework is based on both the best Bayesian bound (MMSE) and a constrained optimization problem. By rewriting the problem of the MMSEE as a continuous constrained optimization problem and by relaxing these constraints, we reobtain the lower bounds of the Weiss-Weinstein family. Moreover, this framework allows us to propose new minimal bounds. In this way we propose an extension of the Bayesian Bhattacharyya bound and a Bayesian version of the Abel bound. Additionally, we give some closed-form expressions of several minimal bounds for both a general Gaussian observation model with parameterized mean and a spectral analysis model.

## VII. Appendix

### A. Proof of Theorem 1

This proof is based on the three following lemmas.

*Lemma 1:*

$$\mathcal{C}_1 = \mathcal{C}_2 \tag{94}$$

*Lemma 2:*

$$\mathcal{C}_1 = \mathcal{C}_3 \tag{95}$$

*Lemma 3:*

$$\mathcal{C}_1 = \mathcal{C}_4 \tag{96}$$

*Proof of Lemma 1:*

- $C_2 \subset C_1$ : we assume that $\forall f(\mathbf{y},\theta) \in \mathcal{F}$, $\int\limits_{\Theta}\int\limits_{\Omega} v(\mathbf{y},\theta) \frac{\partial f(\mathbf{y},\theta)}{\partial\theta} d\mathbf{y} d\theta = \int\limits_{\Theta}\int\limits_{\Omega} f(\mathbf{y},\theta)\, d\mathbf{y} d\theta$. Since,

$$\frac{\partial\left[v(\mathbf{y},\theta) f(\mathbf{y},\theta)\right]}{\partial\theta} = v(\mathbf{y},\theta)\frac{\partial f(\mathbf{y},\theta)}{\partial\theta} + \frac{\partial v(\mathbf{y},\theta)}{\partial\theta} f(\mathbf{y},\theta), \tag{97}$$

we have

$$\int\limits_{\Theta}\int\limits_{\Omega} \frac{\partial\left[v(\mathbf{y},\theta) f(\mathbf{y},\theta)\right]}{\partial\theta} - \frac{\partial v(\mathbf{y},\theta)}{\partial\theta} f(\mathbf{y},\theta)\, d\mathbf{y} d\theta = \int\limits_{\Theta}\int\limits_{\Omega} f(\mathbf{y},\theta)\, d\mathbf{y} d\theta \tag{98}$$

$$\Longrightarrow \int_\Theta \int_\Omega \left(1+\frac{\partial v\left(\mathbf{y},\theta\right)}{\partial \theta}\right) f\left(\mathbf{y},\theta\right) d\mathbf{y} d\theta = 0. \tag{99}$$

Since the expression (99) holds for any $f\left(\mathbf{y},\theta\right)$, if we choose $f\left(\mathbf{y},\theta\right) = 1+\frac{\partial v(\mathbf{y},\theta)}{\partial \theta}$, we obtain

$$\int_\Theta \int_\Omega \left(1+\frac{\partial v\left(\mathbf{y},\theta\right)}{\partial \theta}\right)^2 d\mathbf{y} d\theta = 0 \Longrightarrow 1+\frac{\partial v\left(\mathbf{y},\theta\right)}{\partial \theta} = 0 \Longrightarrow v\left(\mathbf{y},\theta\right) = z\left(\mathbf{y}\right) - \theta, \tag{100}$$

where $z\left(\mathbf{y}\right)$ is a function of $\mathbf{y}$ only.

- $C_1 \subset C_2$ : on the other hand, if we assume that $v\left(\mathbf{y},\theta\right) = z\left(\mathbf{y}\right) - \theta$, then

$$\begin{aligned}\int_\Theta \int_\Omega v\left(\mathbf{y},\theta\right) \frac{\partial f\left(\mathbf{y},\theta\right)}{\partial \theta} d\mathbf{y} d\theta &= \int_\Theta \int_\Omega \left(z\left(\mathbf{y}\right) - \theta\right) \frac{\partial f\left(\mathbf{y},\theta\right)}{\partial \theta} d\mathbf{y} d\theta \\ &= \int_\Theta \int_\Omega \frac{\partial\left[\left(z\left(\mathbf{y}\right) - \theta\right) f\left(\mathbf{y},\theta\right)\right]}{\partial \theta} + f\left(\mathbf{y},\theta\right) d\mathbf{y} d\theta \\ &= \int_\Theta \int_\Omega f\left(\mathbf{y},\theta\right) d\mathbf{y} d\theta \quad \forall f\left(\mathbf{y},\theta\right) \in \mathcal{F} \end{aligned} \tag{101}$$

These two items prove Lemma 1.■

*Proof of Lemma 2:*

- $C_3 \subset C_1$ : we assume that $\forall f\left(\mathbf{y},\theta\right) \in \mathcal{F}$ such that $\int_\Theta \int_\Omega f\left(\mathbf{y},\theta\right) d\mathbf{y} d\theta = 1$ and $\forall h$ such that $\theta + h \in \Theta$,

$$\int_\Theta \int_\Omega v\left(\mathbf{y},\theta\right)\left(f\left(\mathbf{y},\theta+h\right) - f\left(\mathbf{y},\theta\right)\right) d\mathbf{y} d\theta = h. \tag{102}$$

Then, when $h \rightarrow 0$, we have

$$\int_\Theta \int_\Omega v\left(\mathbf{y},\theta\right) \frac{\partial f\left(\mathbf{y},\theta\right)}{\partial \theta} d\mathbf{y} d\theta = 1 \Longrightarrow v\left(\mathbf{y},\theta\right) = z\left(\mathbf{y}\right) - \theta, \tag{103}$$

thanks to the result of the first item of Lemma 1.

- $C_1 \subset C_3$ : on the other hand, if we assume $v\left(\mathbf{y},\theta\right) = z\left(\mathbf{y}\right) - \theta$, then by setting $\varphi = \theta + h$

$$\begin{aligned}\int_\Theta \int_\Omega v\left(\mathbf{y},\theta\right) f\left(\mathbf{y},\theta+h\right) d\mathbf{y} d\theta &= \int_\Theta \int_\Omega \left(z\left(\mathbf{y}\right) - \theta\right) f\left(\mathbf{y},\theta+h\right) d\mathbf{y} d\theta \\ &= \int_\Theta \int_\Omega \left(z\left(\mathbf{y}\right) - \varphi + h\right) f\left(\mathbf{y},\varphi\right) d\mathbf{y} d\varphi \\ &= \int_\Theta \int_\Omega \left(z\left(\mathbf{y}\right) - \varphi\right) f\left(\mathbf{y},\varphi\right) d\mathbf{y} d\varphi + h, \end{aligned} \tag{104}$$

leading to

$$\int_\Theta \int_\Omega \left(z\left(\mathbf{y}\right) - \theta\right)\left(f\left(\mathbf{y},\theta+h\right) - f\left(\mathbf{y},\theta\right)\right) d\mathbf{y} d\theta = h, \tag{105}$$

$\forall f\left(\mathbf{y},\theta\right) \in \mathcal{F}$ such that $\int_\Theta \int_\Omega f\left(\mathbf{y},\theta\right) d\mathbf{y} d\theta = 1$ and $\forall h$ such that $\theta + h \in \Theta$.

These two items prove Lemma 2.■

*Proof of Lemma 3:*

- $C_4 \subset C_1$ : let $L(\mathbf{y},\eta,\theta) = \frac{f(\mathbf{y},\eta)}{f(\mathbf{y},\theta)}$ and assume that $\forall f(\mathbf{y},\theta) \in \mathcal{F}$ such that $\int\limits_{\Theta}\int\limits_{\Omega} f(\mathbf{y},\theta)\, d\mathbf{y}d\theta = 1, \forall h$ such that $\theta \pm h \in \Theta$ and $\forall s \in [0,1]$,

$$\int\limits_{\Theta}\int\limits_{\Omega} v(\mathbf{y},\theta)\left[L^s(\mathbf{y},\theta+h,\theta) - L^{1-s}(\mathbf{y},\theta-h,\theta)\right] f(\mathbf{y},\theta)\, d\mathbf{y}d\theta = h\int\limits_{\Theta}\int\limits_{\Omega} L^{1-s}(\mathbf{y},\theta-h,\theta) f(\mathbf{y},\theta)\, d\mathbf{y}d\theta, \tag{106}$$

Then, when $s \to 1$, we obtain

$$\int\limits_{\Theta}\int\limits_{\Omega} v(\mathbf{y},\theta)\left(f(\mathbf{y},\theta+h) - f(\mathbf{y},\theta)\right) d\mathbf{y}d\theta{=}h \quad \Longrightarrow \quad v(\mathbf{y},\theta) = z(\mathbf{y}) - \theta, \tag{107}$$

thanks to the result of the first item of Lemma 2.

- $C_1 \subset C_4$ : on the other hand, if we assume $v(\mathbf{y},\theta) = z(\mathbf{y}) - \theta$, then by letting $\varphi = \theta + h$

$$\begin{aligned}
\int\limits_{\Theta}\int\limits_{\Omega} v(\mathbf{y},\theta) L^s(\mathbf{y},\theta+h,\theta) f(\mathbf{y},\theta)\, d\theta d\mathbf{y} &= \int\limits_{\Theta}\int\limits_{\Omega} (z(\mathbf{y})-\theta) L^s(\mathbf{y},\theta+h,\theta) f(\mathbf{y},\theta)\, d\theta d\mathbf{y} \\
&= \int\limits_{\Theta}\int\limits_{\Omega} (z(\mathbf{y})-\varphi) L^{1-s}(\mathbf{y},\varphi-h,\varphi) f(\mathbf{y},\varphi)\, d\varphi d\mathbf{y} \\
&\quad + h\int\limits_{\Theta}\int\limits_{\Omega} L^{1-s}(\mathbf{y},\varphi-h,\varphi) f(\mathbf{y},\varphi)\, d\varphi d\mathbf{y},
\end{aligned} \tag{108}$$

leading to

$$\int\limits_{\Theta}\int\limits_{\Omega} v(\mathbf{y},\theta)\left[L^s(\mathbf{y},\theta+h,\theta) - L^{1-s}(\mathbf{y},\theta-h,\theta)\right] f(\mathbf{y},\theta)\, d\theta d\mathbf{y} = h\int\limits_{\Theta}\int\limits_{\Omega} L^{1-s}(\mathbf{y},\theta-h,\theta) f(\mathbf{y},\theta)\, d\theta d\mathbf{y}, \tag{109}$$

$\forall f(\mathbf{y},\theta) \in \mathcal{F}$ such that $\int\limits_{\Theta}\int\limits_{\Omega} f(\mathbf{y},\theta)\, d\mathbf{y}d\theta = 1, \forall h$ such that $\theta \pm h \in \Theta$ and $\forall s \in [0,1]$.

These two items prove Lemma 3.■

Lemmas 1, 2, and 3 prove Theorem 1.■

### *B. Proof of Theorem 2*

Let $\mathcal{U}$ be a vector space of any dimension on the field of real numbers $\mathbb{R}$, with an inner product denoted by $\langle \mathbf{u},\mathbf{w}\rangle$, where $\mathbf{u}$ and $\mathbf{w}$ are two vectors of $\mathcal{U}$. Let $\{\mathbf{g}_1,\ldots,\mathbf{g}_K\}$ be a family of $K$ independent vectors of $\mathcal{U}$ and $\mathbf{c} = \begin{bmatrix} c_1 & \cdots & c_K \end{bmatrix}^T$ be a vector of $\mathbb{R}^K$. We are interested in the solution of the minimization of $\langle \mathbf{u},\mathbf{u}\rangle$ subject to the following $K$ linear constraints $\langle \mathbf{u},\mathbf{g}_k\rangle = c_k$, $k \in [1,K]$.

Let $\mathcal{G}$ be the vectorial sub-space of dimension $K$ generated by the elements $\{\mathbf{g}_1,\ldots,\mathbf{g}_K\}$. Then, $\forall \mathbf{u} \in \mathcal{U}$, $\mathbf{u} = \mathbf{u}_{\mathcal{G}} + d\mathbf{u}$, where $\mathbf{u}_{\mathcal{G}}$ is the orthogonal projection of $\mathbf{u}$ on $\mathcal{G}$, i.e. the vector $\mathbf{u}_{\mathcal{G}} \in \mathcal{G}$ such that $\langle \mathbf{u} - \mathbf{u}_{\mathcal{G}},\mathbf{g}_k\rangle = 0$, $k \in [1,K]$ (see Figure (2) for a graphical representation).

Let $\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 & \cdots & \alpha_K \end{bmatrix}^T$ be the coordinates of $\mathbf{u}_{\mathcal{G}}$ in the basis $\{\mathbf{g}_1,\ldots,\mathbf{g}_K\}$ of $\mathcal{G}$ (i.e., $\mathbf{u}_{\mathcal{G}} = \sum_{k=1}^{K} \alpha_k \mathbf{g}_k$). These coordinates satisfy: $\langle \mathbf{u},\mathbf{g}_k\rangle = \langle \mathbf{u}_{\mathcal{G}},\mathbf{g}_k\rangle$, $k \in [1,K]$. Moreover, if $\mathbf{u}$ satisfies the $K$ constraints $\langle \mathbf{u},\mathbf{g}_k\rangle = c_k$,

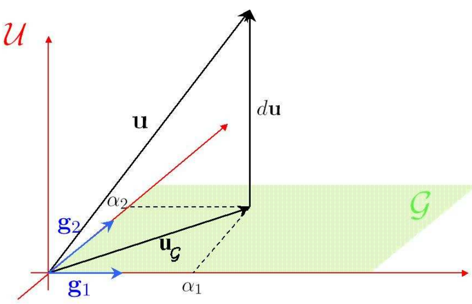


Fig. 2. Graphical representation of the problem

$k \in [1, K]$, then

$$\begin{aligned} \langle \mathbf{u}, \mathbf{g}_k \rangle &= c_k \\ &\Rightarrow \langle \mathbf{u}_{\mathcal{G}}, \mathbf{g}_k \rangle = c_k \\ &\Rightarrow \left\langle \sum_{l=1}^{K} \alpha_l \mathbf{g}_l, \mathbf{g}_k \right\rangle = c_k \\ &\Rightarrow \sum_{l=1}^{K} \alpha_l \langle \mathbf{g}_l, \mathbf{g}_k \rangle = c_k, \end{aligned} \tag{110}$$

i.e., by a matricial rewriting $\mathbf{G}\boldsymbol{\alpha} = \mathbf{c}$, where $\mathbf{G}$ is the Gram matrix associated to the family $\{\mathbf{g}_1, \ldots, \mathbf{g}_K\}$: $G_{k,l} = \langle \mathbf{g}_l, \mathbf{g}_k \rangle$. The equation $\mathbf{G}\boldsymbol{\alpha} = \mathbf{c}$ has for unique solution $\boldsymbol{\alpha} = \mathbf{G}^{-1}\mathbf{c}$. Let $\mathbf{u}_{\mathcal{G},\mathbf{c}}$ be the vector of $\mathcal{G}$ corresponding to this solution. Then, $\forall \mathbf{u} \in \mathcal{U}$ and for satisfying the $K$ aforementioned constraints we have $\langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{u}_{\mathcal{G},\mathbf{c}}, \mathbf{u}_{\mathcal{G},\mathbf{c}} \rangle + \langle d\mathbf{u}, d\mathbf{u} \rangle \geq \langle \mathbf{u}_{\mathcal{G},\mathbf{c}}, \mathbf{u}_{\mathcal{G},\mathbf{c}} \rangle$, and the minimum is achieved for $d\mathbf{u} = 0$, which means that $\mathbf{u}_{\mathcal{G},\mathbf{c}}$ is the solution of the problem. The value of the minimal norm is given by

$$\begin{aligned} \langle \mathbf{u}_{\mathcal{G},\mathbf{c}}, \mathbf{u}_{\mathcal{G},\mathbf{c}} \rangle &= \left\langle \sum_{k=1}^{K} \alpha_k \mathbf{g}_k, \sum_{l=1}^{K} \alpha_l \mathbf{g}_l \right\rangle \\ &= \sum_{k=1}^{K} \sum_{l=1}^{K} \alpha_k \alpha_l \langle \mathbf{g}_k, \mathbf{g}_l \rangle \\ &= \boldsymbol{\alpha}^T \mathbf{G} \boldsymbol{\alpha} \\ &= \left( \mathbf{G}^{-1}\mathbf{c} \right)^T \mathbf{G}\mathbf{G}^{-1}\mathbf{c} \\ &= \mathbf{c}^T \mathbf{G}^{-1} \mathbf{c}. \end{aligned} \tag{111}$$

■

### C. Derivation of the Bayesian Abel bound

We have to calculate the quadratic form $\mathbf{c}^T \mathbf{G}^{-1} \mathbf{c}$ (29). Since

$$\int_{\Theta} \int_{\Omega} p(\mathbf{x}, \theta + h_i)\, d\theta d\mathbf{x} = 1 \quad \forall h_i \in \mathbb{R}, \tag{112}$$

and, due to (9),

$$\int_{\Theta}\int_{\Omega}\frac{\partial^{i}p(\mathbf{x},\theta)}{\partial\theta^{i}}d\theta d\mathbf{x}=0\quad\forall i\geq 1, \tag{113}$$

the matrix $\mathbf{G}=\int_{\Theta}\int_{\Omega}\mathbf{g}\mathbf{g}^{T}d\theta d\mathbf{x}$ can now be written as the following partitioned matrix:

$$\mathbf{G}=\begin{pmatrix}1 & \mathbf{0}_{1\times m} & \mathbf{0}_{1\times r}\\ \mathbf{0}_{m\times 1} & \mathbf{B} & \mathbf{\Gamma}^{T}\\ \mathbf{0}_{r\times 1} & \mathbf{\Gamma} & \mathbf{D}\end{pmatrix}, \tag{114}$$

where the elements $\{\mathbf{B}\}_{i,j}$ and $\{\mathbf{D}\}_{i,j}$ of the matrices $\mathbf{B}$ $(m\times m)$ and $\mathbf{D}$ $(r\times r)$ are given by relation (37) and (48), respectively, and the element $\{\mathbf{\Gamma}\}_{i,j}$ of the matrix $\mathbf{\Gamma}$ $(r\times m)$ is given by

$$\begin{aligned}\{\mathbf{\Gamma}\}_{i,j} &= \int_{\Theta}\int_{\Omega}\frac{p(\mathbf{x},\theta+h_{i})-p(\mathbf{x},\theta)}{p(\mathbf{x},\theta)}\frac{\partial^{j}p(\mathbf{x},\theta)}{\partial\theta^{j}}d\theta d\mathbf{x},\\ &= \int_{\Theta}\int_{\Omega}\frac{p(\mathbf{x},\theta+h_{i})}{p(\mathbf{x},\theta)}\frac{\partial^{j}p(\mathbf{x},\theta)}{\partial\theta^{j}}d\theta d\mathbf{x}.\end{aligned} \tag{115}$$

Let $\tilde{\mathbf{G}}=\begin{pmatrix}\mathbf{B} & \mathbf{\Gamma}^{T}\\ \mathbf{\Gamma} & \mathbf{D}\end{pmatrix}$ and $\mathbf{c}=\left[0\ \boldsymbol{\alpha}^{T}\ \mathbf{h}^{T}\right]^{T}$, where $\boldsymbol{\alpha}=\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix}^{T}$ (size $m\times 1$), and $\mathbf{h}=\begin{bmatrix}h_{1} & \cdots & h_{r}\end{bmatrix}^{T}$. Since the first element of $\mathbf{c}$ is null, only the right bottom corner $\tilde{\mathbf{G}}^{-1}$ (size $(m+r)\times(m+r)$) of $\mathbf{G}^{-1}$ is of interest. $\tilde{\mathbf{G}}^{-1}$ is given straightforwardly by

$$\tilde{\mathbf{G}}^{-1}=\begin{pmatrix}\mathbf{B} & \mathbf{\Gamma}^{T}\\ \mathbf{\Gamma} & \mathbf{D}\end{pmatrix}^{-1}. \tag{116}$$

Consequently, the Bayesian Abel bound denoted $BAB_{m,r}$ is then given by

$$BAB_{m,r}=\left[\boldsymbol{\alpha}^{T}\ \mathbf{h}^{T}\right]\begin{pmatrix}\mathbf{B} & \mathbf{\Gamma}^{T}\\ \mathbf{\Gamma} & \mathbf{D}\end{pmatrix}^{-1}\begin{bmatrix}\boldsymbol{\alpha}\\ \mathbf{h}\end{bmatrix}. \tag{117}$$

After some algebraic effort, we obtain the final form:

$$BAB_{m,r}=\boldsymbol{\alpha}^{T}\mathbf{B}^{-1}\boldsymbol{\alpha}+\mathbf{u}^{T}\mathbf{J}^{-1}\mathbf{u}, \tag{118}$$

with

$$\begin{cases}\mathbf{u}=\mathbf{\Gamma}\mathbf{B}^{-1}\boldsymbol{\alpha}-\mathbf{h},\\ \mathbf{J}=\mathbf{D}-\mathbf{\Gamma}\mathbf{B}^{-1}\mathbf{\Gamma}^{T}.\end{cases} \tag{119}$$

■

*D. Minimal bounds derivation for the Gaussian observation model with parameterized mean*

*1) Bayesian Cramér-Rao bound:* The Bayesian Cramér-Rao bound can be divided into two terms [20]:

$$BCRB=\left(\int_{\Theta}CRB^{-1}(\theta)\,p(\theta)\,d\theta-\int_{\Theta}\frac{\partial^{2}\ln p(\theta)}{\partial\theta^{2}}p(\theta)\,d\theta\right)^{-1}, \tag{120}$$

where $CRB(\theta)$ is the standard (i.e., deterministic) Cramér-Rao bound given by [68]:

$$CRB(\theta_0) = \frac{\sigma^2}{2\left\| \left.\frac{\partial \mathbf{m}(\theta)}{\partial \theta}\right|_{\theta_0} \right\|^2}, \tag{121}$$

where $\theta_0$ is the true value of the parameter in the deterministic context.

The second term of (120) is

$$\begin{aligned}\int_{\Theta} \frac{\partial^2 \ln p(\theta)}{\partial \theta^2} p(\theta)\, d\theta &= -\frac{1}{2\sigma_\theta^2} \int_{\Theta} \frac{\partial^2 (\theta - \mu)^2}{\partial \theta^2} p(\theta)\, d\theta \\ &= -\frac{1}{\sigma_\theta^2} \int_{\Theta} p(\theta)\, d\theta = -\frac{1}{\sigma_\theta^2}. \end{aligned} \tag{122}$$

Consequently,

$$BCRB = \frac{\sigma_\theta^2}{\frac{2\sigma_\theta^2}{\sigma^2} \mathbb{E}_\theta \left[ \left\| \frac{\partial \mathbf{m}(\theta)}{\partial \theta} \right\|^2 \right] + 1}. \tag{123}$$

*2) Bobrovsky-Zakaï bound:* The Bobrovsky-Zakaï bound is given by

$$BZB = \sup_h \frac{h^2}{\int_\Theta \int_\Omega \frac{p^2(\mathbf{y},\theta+h)}{p(\mathbf{y},\theta)} d\mathbf{y} d\theta - 1}. \tag{124}$$

The double integral in the last equation can be rewritten as follows:

$$\int_\Theta \int_\Omega \frac{p^2(\mathbf{y},\theta+h)}{p(\mathbf{y},\theta)} d\mathbf{y} d\theta = \int_\Theta \frac{p^2(\theta+h)}{p(\theta)} \int_\Omega \frac{p^2(\mathbf{y}|\,\theta+h)}{p(\mathbf{y}|\,\theta)} d\mathbf{y} d\theta. \tag{125}$$

The term $\frac{p^2(\mathbf{y}|\theta+h)}{p(\mathbf{y}|\theta)}$ becomes

$$\begin{aligned}\frac{p^2(\mathbf{y}|\,\theta+h)}{p(\mathbf{y}|\,\theta)} &= \frac{1}{(\pi\sigma^2)^N} e^{-\frac{1}{\sigma^2}\left(2(\mathbf{y}-\mathbf{m}(\theta+h))^H(\mathbf{y}-\mathbf{m}(\theta+h)) - (\mathbf{y}-\mathbf{m}(\theta))^H(\mathbf{y}-\mathbf{m}(\theta))\right)} \\ &= \frac{1}{(\pi\sigma^2)^N} e^{-\frac{1}{\sigma^2}\left(\|\mathbf{y}\|^2 + 2\|\mathbf{m}(\theta+h)\|^2 - \|\mathbf{m}(\theta)\|^2 - 2\,\mathrm{Re}\left\{\mathbf{y}^H(2\mathbf{m}(\theta+h) - \mathbf{m}(\theta))\right\}\right)}. \end{aligned} \tag{126}$$

Let $\mathbf{x} = \mathbf{y} - 2\mathbf{m}(\theta+h) + \mathbf{m}(\theta)$, and note that

$$\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2 + \|2\mathbf{m}(\theta+h) - \mathbf{m}(\theta)\|^2 - 2\,\mathrm{Re}\left\{\mathbf{y}^H(2\mathbf{m}(\theta+h) - \mathbf{m}(\theta))\right\}.$$

Consequently,

$$\begin{aligned}\int_\Omega \frac{p^2(\mathbf{y}|\,\theta+h)}{p(\mathbf{y}|\,\theta)} d\mathbf{y} &= \frac{1}{(\pi\sigma^2)^N} \int_\Omega e^{-\frac{1}{\sigma^2}\left(\|\mathbf{x}\|^2 + 2\|\mathbf{m}(\theta+h)\|^2 - \|\mathbf{m}(\theta)\|^2 - \|2\mathbf{m}(\theta+h) - \mathbf{m}(\theta)\|^2\right)} d\mathbf{x} \\ &= \frac{1}{(\pi\sigma^2)^N} e^{-\frac{1}{\sigma^2}\left(2\|\mathbf{m}(\theta+h)\|^2 + \|\mathbf{m}(\theta)\|^2 - \|2\mathbf{m}(\theta+h) - \mathbf{m}(\theta)\|^2\right)} \underbrace{\int_\Omega e^{-\frac{1}{\sigma^2}\|\mathbf{x}\|^2} d\mathbf{x}}_{=(\pi\sigma^2)^N} \\ &= e^{\frac{2}{\sigma^2}\|\mathbf{m}(\theta+h) - \mathbf{m}(\theta)\|^2}. \end{aligned} \tag{127}$$

The Bobrovsky-Zakaï bound is finally given by

$$BZB = \sup_h \frac{h^2}{\int_\Theta \frac{p^2(\theta+h)}{p(\theta)} e^{\frac{2}{\sigma^2}\|\mathbf{m}(\theta+h) - \mathbf{m}(\theta)\|^2} d\theta - 1}. \tag{128}$$

*3) Bayesian Abel bound:* We have to calculate

$$\begin{aligned}\phi(h) &= \frac{1}{h}\int_{\Theta}\int_{\Omega}\frac{\partial\ln p(\mathbf{y},\theta)}{\partial\theta}p(\mathbf{y},\theta+h)\,d\mathbf{y}d\theta\\ &= \frac{1}{h}\int_{\Theta}p(\theta+h)\int_{\Omega}\left(\frac{\partial\ln p(\mathbf{y}|\,\theta)+\ln p(\theta)}{\partial\theta}\right)p(\mathbf{y}|\,\theta+h)\,d\mathbf{y}d\theta\\ &= \frac{1}{h}\int_{\Theta}p(\theta+h)\int_{\Omega}\frac{\partial\ln p(\mathbf{y}|\,\theta)}{\partial\theta}p(\mathbf{y}|\,\theta+h)\,d\mathbf{y}d\theta+\frac{1}{h}\int_{\Theta}\frac{\partial\ln p(\theta)}{\partial\theta}p(\theta+h)\,d\theta. \end{aligned} \tag{129}$$

The first term in (129) is given by

$$\begin{aligned}\int_{\Omega}\frac{\partial\ln p(\mathbf{y}|\,\theta)}{\partial\theta}p(\mathbf{y}|\,\theta+h)\,d\mathbf{y} &= -\frac{1}{\sigma^2}\int_{\Omega}\frac{\partial(\mathbf{y}-\mathbf{m}(\theta))^H(\mathbf{y}-\mathbf{m}(\theta))}{\partial\theta}p(\mathbf{y}|\,\theta+h)\,d\mathbf{y}\\ &= \frac{2}{\sigma^2}\int_{\Omega}\mathrm{Re}\left\{\frac{\partial\mathbf{m}^H(\theta)}{\partial\theta}(\mathbf{y}-\mathbf{m}(\theta))\right\}p(\mathbf{y}|\,\theta+h)\,d\mathbf{y}\\ &= \frac{2}{\sigma^2}\mathrm{Re}\left\{\frac{\partial\mathbf{m}^H(\theta)}{\partial\theta}\left(\int_{\Omega}\mathbf{y}p(\mathbf{y}|\,\theta+h)\,d\mathbf{y}-\mathbf{m}(\theta)\right)\right\}\\ &= \frac{2}{\sigma^2}\mathrm{Re}\left\{\frac{\partial\mathbf{m}^H(\theta)}{\partial\theta}(\mathbf{m}(\theta+h)-\mathbf{m}(\theta))\right\}. \end{aligned} \tag{130}$$

For the second term in (129), we have

$$\begin{aligned}\frac{1}{h}\int_{\Theta}p(\theta+h)\frac{\partial\ln p(\theta)}{\partial\theta}d\theta &= -\frac{1}{h\sigma_\theta^2}\int_{\Theta}(\theta-\mu)\,p(\theta+h)\,d\theta\\ &= \frac{1}{\sigma_\theta^2}. \end{aligned} \tag{131}$$

Finally:

$$\phi(h)=\frac{1}{\sigma_\theta^2}+\frac{2}{h\sigma^2}\mathbb{E}_{\theta+h}\left[\mathrm{Re}\left\{\frac{\partial\mathbf{m}^H(\theta)}{\partial\theta}(\mathbf{m}(\theta+h)-\mathbf{m}(\theta))\right\}\right]. \tag{132}$$

*4) Weiss-Weinstein bound:* We have to calculate

$$\eta(\alpha,\beta)=\ln\int_{\Theta}\int_{\Omega}\frac{p^{\alpha}(\mathbf{y},\theta+\beta)}{p^{\alpha-1}(\mathbf{y},\theta)}d\mathbf{y}d\theta. \tag{133}$$

This function can be modified as follows

$$\eta(\alpha,\beta)=\ln\int_{\Theta}\frac{p^{\alpha}(\theta+\beta)}{p^{\alpha-1}(\theta)}\int_{\Omega}\frac{p^{\alpha}(\mathbf{y}|\,\theta+\beta)}{p^{\alpha-1}(\mathbf{y}|\,\theta)}d\mathbf{y}d\theta. \tag{134}$$

Let us first study the term

$$\begin{aligned}\frac{p^{\alpha}(\mathbf{y}|\,\theta+\beta)}{p^{\alpha-1}(\mathbf{y}|\,\theta)} &= \frac{1}{(\pi\sigma^2)^N}e^{-\frac{1}{\sigma^2}\left(\alpha(\mathbf{y}-\mathbf{m}(\theta+\beta))^H(\mathbf{y}-\mathbf{m}(\theta+\beta))-(\alpha-1)(\mathbf{y}-\mathbf{m}(\theta))^H(\mathbf{y}-\mathbf{m}(\theta))\right)}\\ &= \frac{1}{(\pi\sigma^2)^N}e^{-\frac{1}{\sigma^2}\left(\|\mathbf{y}\|^2+\alpha\|\mathbf{m}(\theta+\beta)\|^2-(\alpha-1)\|\mathbf{m}(\theta)\|^2-2\,\mathrm{Re}\left\{\mathbf{y}^H(\alpha\mathbf{m}(\theta+\beta)-(\alpha-1)\mathbf{m}(\theta))\right\}\right)} \end{aligned} \tag{135}$$

Let $\mathbf{x}=\mathbf{y}-(\alpha\mathbf{m}(\theta+\beta)-(\alpha-1)\mathbf{m}(\theta))$. Note that

$$\|\mathbf{x}\|^2=\|\mathbf{y}\|^2+\|\alpha\mathbf{m}(\theta+\beta)-(\alpha-1)\mathbf{m}(\theta)\|^2-2\,\mathrm{Re}\left\{\mathbf{y}^H(\alpha\mathbf{m}(\theta+\beta)-(\alpha-1)\mathbf{m}(\theta))\right\}. \tag{136}$$

Consequently,

$$\begin{aligned}\int\limits_{\Omega}\frac{p^{\alpha}\left(\mathbf{y}|\,\theta+\beta\right)}{p^{\alpha-1}\left(\mathbf{y}|\,\theta\right)}d\mathbf{y} &= \frac{1}{\left(\pi\sigma^{2}\right)^{N}}\int\limits_{\Omega}e^{-\frac{1}{\sigma^{2}}\left(\|\mathbf{x}\|^{2}-\|\alpha\mathbf{m}(\theta+\beta)-(\alpha-1)\mathbf{m}(\theta)\|^{2}+\alpha\|\mathbf{m}(\theta+\beta)\|^{2}-(\alpha-1)\|\mathbf{m}(\theta)\|^{2}\right)}d\mathbf{x}\\ &= \frac{1}{\left(\pi\sigma^{2}\right)^{N}}e^{-\frac{1}{\sigma^{2}}\left(-\|\alpha\mathbf{m}(\theta+\beta)-(\alpha-1)\mathbf{m}(\theta)\|^{2}+\alpha\|\mathbf{m}(\theta+\beta)\|^{2}-(\alpha-1)\|\mathbf{m}(\theta)\|^{2}\right)}\underbrace{\int\limits_{\Omega}e^{-\frac{\|\mathbf{x}\|^{2}}{\sigma^{2}}}d\mathbf{x}}_{=\left(\pi\sigma^{2}\right)^{N}}\\ &= e^{\frac{\alpha(\alpha-1)}{\sigma^{2}}\|\mathbf{m}(\theta+\beta)-\mathbf{m}(\theta)\|^{2}}. \end{aligned} \tag{137}$$

For the second term,

$$\frac{p^{\alpha}\left(\theta+\beta\right)}{p^{\alpha-1}\left(\theta\right)}=\frac{1}{\sqrt{2\pi}\sigma_{\theta}}e^{-\frac{1}{2\sigma_{\theta}^{2}}\left[\alpha(\theta+\beta-\mu)^{2}-(\alpha-1)(\theta-\mu)^{2}\right]}. \tag{138}$$

Finally, the semi-invariant moment generating function is given by

$$\eta\left(\alpha,\beta\right)=\ln\frac{1}{\sqrt{2\pi}\sigma_{\theta}}\int\limits_{\Theta}e^{\frac{\alpha(\alpha-1)}{\sigma^{2}}\|\mathbf{m}(\theta+\beta)-\mathbf{m}(\theta)\|^{2}-\frac{1}{2\sigma_{\theta}^{2}}\left(\theta-\left(\sqrt{\alpha(\alpha-1)}-\alpha\right)h-\mu\right)\left(\theta+\left(\sqrt{\alpha(\alpha-1)}+\alpha\right)h-\mu\right)}d\theta. \tag{139}$$

### *E. Bayesian bounds derivation for a spectral analysis problem*

*1) Cramér-Rao bound:* The Bayesian Cramér-Rao bound is given by (123)

$$BCRB=\frac{\sigma_{\theta}^{2}}{\frac{2\sigma_{\theta}^{2}}{\sigma^{2}}\mathbb{E}_{\theta}\left[\left\|\frac{\partial\mathbf{m}(\theta)}{\partial\theta}\right\|^{2}\right]+1}. \tag{140}$$

The term $\left\|\frac{\partial\mathbf{m}(\theta)}{\partial\theta}\right\|^{2}$ can be written

$$\begin{aligned}\left\|\frac{\partial\mathbf{m}\left(\theta\right)}{\partial\theta}\right\|^{2} &= \left\|a\frac{\partial\mathbf{s}\left(\theta\right)}{\partial\theta}\right\|^{2}=\sum_{k=0}^{N-1}a^{2}\left(j2\pi ke^{j2\pi k\theta}\right)\left(-j2\pi ke^{-j2\pi k\theta}\right)\\ &= a^{2}4\pi^{2}\sum_{k=0}^{N-1}k^{2}=\frac{2\left(a\pi\right)^{2}}{3}N(2N-1)(N-1), \end{aligned} \tag{141}$$

which is independent of $\theta$. Consequently, the Bayesian Cramér-Rao bound is

$$BCRB=\frac{\sigma_{\theta}^{2}}{SNR\frac{4\pi^{2}\sigma_{\theta}^{2}}{3}N(2N-1)(N-1)+1}. \tag{142}$$

*2) Bobrovsky-Zakaï bound:* The Bobrovsky-Zakaï bound is given by (128)

$$BZB=\sup_{h}\frac{h^{2}}{\int\limits_{\Theta}\frac{p^{2}(\theta+h)}{p(\theta)}e^{\frac{2}{\sigma^{2}}\|\mathbf{m}(\theta+h)-\mathbf{m}(\theta)\|^{2}}d\theta-1}. \tag{143}$$

In the case of our specific model (79), the term $\|\mathbf{m}\left(\theta+h\right)-\mathbf{m}\left(\theta\right)\|^{2}$ can be written

$$\begin{aligned}\|\mathbf{m}\left(\theta+h\right)-\mathbf{m}\left(\theta\right)\|^{2} &= a^{2}\sum_{k=0}^{N-1}\left(e^{j2\pi k(\theta+h)}-e^{j2\pi k\theta}\right)\left(e^{-j2\pi k(\theta+h)}-e^{-j2\pi k\theta}\right)\\ &= a^{2}\sum_{k=0}^{N-1}\left(2-2\operatorname{Re}\left\{e^{j2\pi kh}\right\}\right)=2a^{2}\sum_{k=0}^{N-1}1-\cos\left(2\pi kh\right)\\ &= 2a^{2}\left(N-\sin^{2}\left(\pi hN\right)-\frac{1}{2}\frac{\sin\left(2\pi hN\right)}{\tan\left(\pi h\right)}\right), \end{aligned} \tag{144}$$

which is independent of $\theta$. The term $\int\limits_{\Theta} \frac{p^2(\theta+h)}{p(\theta)} d\theta$ becomes

$$\begin{aligned}\int\limits_{\Theta} \frac{p^2(\theta+h)}{p(\theta)} d\theta &= \frac{1}{\sqrt{2\pi}\sigma_\theta} e^{-\frac{1}{2\sigma_\theta^2}\left[2h^2-4h\mu+\mu^2\right]} \int\limits_{\Theta} e^{-\frac{1}{2\sigma_\theta^2}\left[\theta^2+2\theta(2h-\mu)\right]} d\theta \\ &= e^{-\frac{1}{2\sigma_\theta^2}\left[2h^2-4h\mu+\mu^2\right]+\frac{(2h-\mu)^2}{2\sigma_\theta^2}} = e^{\frac{h^2}{\sigma_\theta^2}}, \end{aligned} \tag{145}$$

where the term $\int\limits_{\Theta} e^{-\frac{1}{2\sigma_\theta^2}\left[\theta^2+2\theta(2h-\mu)\right]} d\theta$ is given by [71] page 355, equation (BI((28))(1),

$$\int\limits_{-\infty}^{\infty} e^{-p^2x^2 \pm qx} dx = \frac{\sqrt{\pi}}{abs(p)} e^{\frac{q^2}{4p^2}}. \tag{146}$$

Finally, the Bobrovsky-Zakaï is given by

$$BZB = \sup_h \frac{h^2}{e^{4SNR\left(N-\sin^2(\pi hN)-\frac{1}{2}\frac{\sin(2\pi hN)}{\tan(\pi h)}\right)+\frac{h^2}{\sigma_\theta^2}} - 1} \tag{147}$$

*3) Bayesian Abel bound:* We have to calculate (132)

$$\phi(h) = \frac{1}{\sigma_\theta^2} + \frac{2}{h\sigma^2} \mathbb{E}_{\theta+h}\left[\mathrm{Re}\left\{\frac{\partial \mathbf{m}^H(\theta)}{\partial \theta}\left(\mathbf{m}(\theta+h)-\mathbf{m}(\theta)\right)\right\}\right]. \tag{148}$$

The term $\mathrm{Re}\left\{\frac{\partial \mathbf{m}^H(\theta)}{\partial \theta}\left(\mathbf{m}(\theta+h)-\mathbf{m}(\theta)\right)\right\}$ can be rewritten as follows:

$$\begin{aligned}\mathrm{Re}\left\{\frac{\partial \mathbf{m}^H(\theta)}{\partial \theta}\left(\mathbf{m}(\theta+h)-\mathbf{m}(\theta)\right)\right\} &= \mathrm{Re}\left\{a^2 \frac{\partial \mathbf{s}^H(\theta)}{\partial \theta}\left(\mathbf{s}(\theta+h)-\mathbf{s}(\theta)\right)\right\} \\ &= a^2\, \mathrm{Re}\left\{\sum_{k=0}^{N-1}\left(e^{j2\pi k(\theta+h)} - e^{j2\pi k\theta}\right)\frac{\partial e^{-j2\pi k\theta}}{\partial \theta}\right\} \\ &= -2\pi a^2\, \mathrm{Re}\left\{\sum_{k=0}^{N-1} jk\left(e^{j2\pi kh}-1\right)\right\} \\ &= 2\pi a^2 \sum_{k=0}^{N-1} k \sin(2\pi kh) \\ &= \pi a^2\left(N\frac{\cos(2\pi hN)}{\tan(\pi h)} - \sin(2\pi hN)\left(\frac{1}{2\sin(\pi h)}+N\right)\right), \end{aligned} \tag{149}$$

which is independent of $\theta$. Consequently,

$$\phi(h) = \frac{1}{\sigma_\theta^2} + \frac{2\pi SNR}{h}\left(N\frac{\cos(2\pi hN)}{\tan(\pi h)} - \sin(2\pi hN)\left(\frac{1}{2\sin(\pi h)}+N\right)\right). \tag{150}$$

*4) Weiss-Weinstein bound:* We have to calculate (139),

$$\begin{aligned}\eta(\alpha,\beta) &= \ln \int\limits_{\Theta} \frac{p^\alpha(\theta+\beta)}{p^{\alpha-1}(\theta)} e^{\frac{\alpha(\alpha-1)}{\sigma^2}\left\|\mathbf{m}(\theta+\beta)-\mathbf{m}(\theta)\right\|^2} d\theta \\ &= \ln\left(e^{2\alpha(\alpha-1)SNR\left(N-\sin^2(\pi\beta N)-\frac{1}{2}\frac{\sin(2\pi\beta N)}{\tan(\pi\beta)}\right)} \int\limits_{\Theta} \frac{p^\alpha(\theta+\beta)}{p^{\alpha-1}(\theta)} d\theta\right), \end{aligned} \tag{151}$$

thanks to (144) and to the independence of $\theta$ in the term $\left\|\mathbf{m}(\theta+\beta)-\mathbf{m}(\theta)\right\|^2$.

The remaining term is given by

$$\begin{aligned}\int_{\Theta}\frac{p^{\alpha}\left(\theta+\beta\right)}{p^{\alpha-1}\left(\theta\right)}d\theta &= \frac{1}{\sqrt{2\pi}\sigma_{\theta}}e^{-\frac{1}{2\sigma_{\theta}^{2}}\left[\alpha\beta^{2}-2\alpha\beta\mu+\mu^{2}\right]}\int_{\Theta}e^{-\frac{1}{2\sigma_{\theta}^{2}}\left[\theta^{2}+2\theta(\alpha\beta-\mu)\right]}d\theta \\ &= e^{-\frac{1}{2\sigma_{\theta}^{2}}\left[\alpha\beta^{2}-2\alpha\beta\mu+\mu^{2}\right]+\frac{(\alpha\beta-\mu)^{2}}{2\sigma_{\theta}^{2}}}=e^{-\frac{\alpha\beta^{2}}{2\sigma_{\theta}^{2}}(1-\alpha)},\end{aligned} \tag{152}$$

where $\int_{\Theta}e^{-\frac{1}{2\sigma_{\theta}^{2}}\left[\theta^{2}+2\theta(\alpha\beta-m)\right]}d\theta$ is obtained thanks to [71] page 355, Equation (BI((28))(1).

Consequently, $\eta\left(\alpha,\beta\right)$ is given by

$$\eta\left(\alpha,\beta\right)=\alpha\left(\alpha-1\right)\left(2SNR\left(N-\sin^{2}\left(\pi\beta N\right)-\frac{1}{2}\frac{\sin\left(2\pi\beta N\right)}{\tan\left(\pi\beta\right)}\right)-\frac{\beta^{2}}{2\sigma_{\theta}^{2}}\right). \tag{153}$$

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