Conditional and Unconditional Cramér–Rao Bounds for Near-Field Source Localization

Mohammed Nabil El Korso, Rémy Boyer, Alexandre Renaux, and Sylvie Marcos

***Abstract*—Near-field source localization problem by a passive antenna array makes the assumption that the time-varying sources are located near the antenna. In this context, the far-field assumption (i.e., planar wavefront) is, of course, no longer valid and one has to consider a more complicated model parameterized by the bearing (as in the far-field case) and by the distance, named range, between the source and a reference coordinate system. One can find a plethora of estimation schemes in the literature, but their ultimate performance in terms of mean square error (MSE) have not been fully investigated. To characterize these performance, the Cramér–Rao bound (CRB) is a popular mathematical tool in signal processing. The main cause for this is that the MSE of several high-resolution direction of arrival algorithms are known to achieve the CRB under quite general/weak conditions. In this correspondence, we derive and analyze the so-called conditional and unconditional CRBs for a single time-varying near-field source. In each case, we obtain non-matrix closed-form expressions. Our approach has two advantages: i) due to the fact that one has to inverse the Fisher information matrix, the computational cost for a large number of snapshots (in the case of the conditional CRB) and/or for a large number of sensors (in the case of the unconditional CRB), of a matrix-based CRB can be high while our approach is low and ii) some useful information can be deduced from the behavior of the bound. In particular, an explicit relationship between the conditional and the unconditional CRBs is provided and one shows that closer is the source from the array and/or higher is the signal carrier frequency, better is the range estimation.**

***Index Terms*—Bearing and range estimation, Cramér–Rao bound, near field, performance analysis, performance bound, source localization.**

I. Introduction

Passive sources localization by an array of sensors is an important topic with a large number of applications, such as sonar, seismology, digital communications, etc. Particularly, the context of far-field sources has been widely investigated in the literature and several algorithms to estimate the localization parameters have been proposed [2]. In this case, the sources are assumed to be far from the array of sensors. Consequently, the propagating waves are assumed to have planar wavefronts when they reach the array. However, when the sources are located in the so-called near-field region, the curvature of the waves impinging on the sensors can no longer be neglected. Therefore, in this scenario, each source is characterized by its bearing and its range.

In array processing, there exist two different models depending on the assumptions about the signal sources: 1) the so-called conditional model, i.e., when the signals are assumed to be deterministic but unknown and 2) the so-called unconditional model, i.e., when the signals are assumed to be driven by a Gaussian random process. Each model is appropriate for a given situation. For example, the assumption of Gaussian source signal is not realistic for several applications (for example, in radar [3] or radio communication applications [4]). A legitimate choice is then to assume that the emitted signals are deterministic and unknown. On the other hand, in some applications it is appropriate to model the sources as stationary Gaussian processes (for examples in seismology and tomography, see [5]). One can find many estimation schemes adapted to near-field source localization (e.g., [6]–[8]), but only a few number of works studying the optimal performance associated with this model have been proposed. To characterize the performance of an estimator in terms of mean square error (MSE), the Cramér–Rao bound (CRB) is certainly the most popular tool [9].

Since, in array processing, two signals models are generally used, it exists two distinct CRB named the Unconditional CRB (UCRB) and the Conditional CRB (CCRB). More precisely, the UCRB is achieved asymptotically, i.e., for a large number of snapshots, by the Unconditional Maximum Likelihood (UML) estimator [10], whereas the CCRB is achieved asymptotically, i.e., at high signal-to-noise ratio, by the Conditional Maximum Likelihood (CML) estimator [11].

Most of the results concerning the UCRB and the CCRB available in the literature deal with the far-field case. Moreover, in some works, only closed-form expressions of the Fisher information matrix are given. We call these cases *matrix expression* of the CRB since the inversion of the FIM is not presented. On the other hand, we will refer to a *non-matrix* expression of the CRB when the inversion of the FIM is proposed. Note that, in the conditional signal model case, this distinction is fundamental since the size of the parameter vector grows with the number of snapshots.

In [12], the UCRB was indirectly derived as the asymptotic, in terms of number of snapshots, covariance matrix of the UML estimator. Ten years after, Stoica *et al.* [13], Pesavento and Gershman [14] and Gershman *et al.* [15] provided a direct (but similar) matrix-based derivation of this bound using the extended Slepian–Bangs formula for a uniform, a nonuniform, and an unknown noise field, respectively. On the other hand, a matrix-based expression of the CCRB for the far-field case was derived by Stoica *et al.* in [16].

Unlike the far-field case, the CRB for the near-field localization problem has been less studied. One can find in [17] matrix-based expressions of the UCRB for range and bearing estimation. Ottersten *et al.* derived a general matrix-based expressions of the UCRB for unknown parameters associated with the emitted signal [10]. Recently, Grosicki *et al.* [6] extended, to the near-field case, the matrix-form expression for the UCRB similar to that given in [12] in the far-field case. Again, one should note that all the closed-form expressions, given in the literature and above concerning the near-field case, are matrix-based expressions stopped before the inversion of the Fisher information matrix. To the best of our knowledge, no non-matrix expressions are available concerning the CCRB and UCRB for range and bearing estimation in the near-field context. The goal of this correspondence is to fill this lack. Particularly, non-matrix closed-form expressions of the CRB in the case of a single deterministic (but unknown) and stochastic time-varying narrowband source in the near-field region are derived and analyzed. Consequently, this approach avoids the costly computational cost of the matrix-based CRB expressions particularly for a large number of snapshots (for the CCRB) and/or for a large number of sensors (for the UCRB). However, it is not the only reason concerning the usefulness of these non-matrix expressions. Deriving non-matrix expressions of the CRB enables us

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The authors are with Laboratoire des Signaux et Systémes (L2S), Université Paris-Sud XI, CNRS, SUPELEC, Gif-Sur-Yvette, 91192, France (e-mail: elkorso@lss.supelec.fr; remy.boyer@lss.supelec.fr; alexandre.renaux@lss.supelec.fr; marcos@lss.supelec.fr).

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to characterize the performance of any unbiased estimator and to use it to deduce some useful information describing the behavior of the MLE variance as a function of the physical parameters.

This correspondence is organized as follows. Section II formulates the problem and basic assumptions. In Section III we present our derivation of the CCRB and the UCRB in the near-field region. Section IV is devoted to the analytical and numerical analysis of the CRB where we provide a discussion on the CRB's behavior. Furthermore, simulation results are provided to validate this theoretical analysis. Finally, conclusions are given in Section V.

Glossary of Notation: The following notations are used through the correspondence. Matrices and vectors are represented by bold uppercase and bold lowercase characters, respectively. Vectors are, by default, in column orientation, whereas $\mathbf{Z}^T, \mathbf{Z}^*, \mathbf{Z}^H, \mathrm{tr}\{\mathbf{Z}\}$ and $\det\{\mathbf{Z}\}$ denote the transpose, the conjugate, the conjugate transpose, the trace and the determinant of the matrix $\mathbf{Z}$, respectively. $[\mathbf{z}]_i$ and $[\mathbf{Z}]_{i,k}$ denote the ith element of the vector $\mathbf{z}$ and the ith row and the kth column element of the matrix $\mathbf{Z}$, respectively. Furthermore, $\Re\{\cdot\}, E\{\cdot\}, \odot, \otimes, \mathrm{diag}(.), \mathrm{bdiag}(.), \mathrm{vec}(.), \delta(.)$ and $\mathrm{mod}(.)$ stand for the real part, the expectation, the Hadamard product, the Kronecker product, the diagonal operator, the block diagonal operator, the vec-operator, the Kronecker symbol and the modulo operator, respectively. $\mathbf{1}_L$ and $\boldsymbol{I}_L$ denote the vector of dimension $L \times 1$ filled by ones and the identity matrix of size $L \times L$, respectively. Finally $j, O(\nu)$ and $\|\boldsymbol{\alpha_i}\|^2 = (1/L)\sum_{t=1}^{L} \alpha_i^2(t)$ denote the complex number $\sqrt{-1}$, the terms of order larger or equal to ν and the normalized norm of the vector $\boldsymbol{\alpha_i}$.

II. Problem Setup and Assumptions

Consider an uniform linear array (ULA) of N sensors with inter-element spacing d that receives a signal emitted by a single near-field and narrowband source. Consequently, the observation model is as follows:

$$x_n(t) = s(t)e^{j2\pi\tau_n} + v_n(t), \quad t = 1, \ldots, L, \quad n = 0, \ldots, N-1$$

where $x_n(t)$ is the observed signal at the output of the $(n+1)$th sensor. In the conditional case, $s(t) = \alpha(t)e^{j(2\pi f_0 t + \psi(t))}$ is the source signal with a carrier frequency equals to f_0 where $\alpha(t)$ and $\psi(t)$ are the real amplitude and the shift phase, respectively. The random process $v_n(t)$ is an additive noise and L is the number of snapshots. The time delay τ_n associated with the signal propagation time from the first sensor to the $(n+1)$th sensor is given by [6]

$$\tau_n = \frac{r}{2\pi\lambda}\left(\sqrt{1 + \frac{n^2d^2}{r^2} - \frac{2nd\sin\theta}{r}} - 1\right)$$

where λ is the signal wavelength and where r and $\theta \in [0, \pi/2]$ denote the range and the bearing of the source, respectively. It is well known that, if the source range is inside of the so-called Fresnel region [7], i.e.,

$$0.62\left(d^3\frac{(N-1)^3}{\lambda}\right)^{1/2} < r < 2d^2\frac{(N-1)^2}{\lambda}, \tag{1}$$

then the time delay τ_n can be approximated by $\tau_n = (\omega n + \phi n^2)/(2\pi) + O(d^2/r^2)$. ω and ϕ are the so-called electric angles which are connected to the physical parameters of the problem by: $\omega = -2\pi d\sin(\theta)/\lambda$ and $\phi = \pi d^2\cos^2(\theta)/(\lambda r)$. Then, neglecting $O(d^2/r^2)$ in the time delay expression [7], the observation model becomes $x_n(t) = s(t)e^{j(\omega n + \phi n^2)} + v_n(t)$. Consequently, the observation vector can be expressed as

$$\boldsymbol{x}(t) = \mathbf{a}(\omega, \phi)s(t) + \mathbf{v}(t) \tag{2}$$

where $\boldsymbol{x}(t) = [x_0(t)\ldots x_{N-1}(t)]^T$, $\mathbf{v}(t) = [v_0(t)\ldots v_{N-1}(t)]^T$ and where the $(n+1)$th element of the steering vector $\mathbf{a}(\omega,\phi)$ is given by $[\mathbf{a}(\omega,\phi)]_{n+1} = e^{j(\omega n + \phi n^2)}$. The noise will be assumed to be a complex circular white Gaussian random process with zero-mean and unknown variance σ^2, uncorrelated both temporally and spatially. Consequently, the joint probability density function of the observations $\chi = [\boldsymbol{x}^T(1)\cdots\boldsymbol{x}^T(L)]^T$ given a parameter vector $\boldsymbol{\eta}$ is given by

$$p(\chi|\boldsymbol{\eta}) = \prod_{t=1}^{L} p(\boldsymbol{x}(t)|\boldsymbol{\eta}) = \frac{1}{\pi^{NL}\det\{\mathbf{R}\}}e^{-(\chi-\boldsymbol{\mu})^H\mathbf{R}^{-1}(\chi-\boldsymbol{\mu})}$$

where $\mathbf{R}$ and $\boldsymbol{\mu}$ denote the covariance matrix and the average of χ, respectively.

III. Cramér–Rao Bounds Derivation

The goal of this section is to derive the CCRB and the UCRB with respect to the bearing and the range. Let $E\{(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta})(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta})^T\}$ be the covariance matrix of an unbiased estimator, $\hat{\boldsymbol{\eta}}$, of a deterministic parameter vector $\boldsymbol{\eta}$. The covariance inequality principle states that, under quite general/weak conditions, the variance satisfies $\mathrm{MSE}([\hat{\boldsymbol{\eta}}]_i) = E\{([\hat{\boldsymbol{\eta}}]_i - [\boldsymbol{\eta}]_i)^2\} \geq [\mathbf{CRB}(\boldsymbol{\eta})]_{i,i}$ where $\mathbf{CRB}(\boldsymbol{\eta}) = \mathbf{FIM}^{-1}(\boldsymbol{\eta})$. In the following, for sake of simplicity the notation, $\mathrm{CRB}([\boldsymbol{\eta}]_i)$ will be used instead of $[\mathbf{CRB}(\boldsymbol{\eta})]_{i,i}$. Since we are working with a complex circular Gaussian observation model, the (ith, kth) element of the Fisher information matrix (FIM) for the parameter vector $\boldsymbol{\eta}$ is well known and can be written as [18]

$$[\mathbf{FIM}(\boldsymbol{\eta})]_{i,k} = \mathrm{tr}\left\{\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial[\boldsymbol{\eta}]_i}\mathbf{R}^{-1}\frac{\partial\mathbf{R}}{\partial[\boldsymbol{\eta}]_k}\right\} + 2\Re\left\{\frac{\partial\boldsymbol{\mu}^H}{\partial[\boldsymbol{\eta}]_i}\mathbf{R}^{-1}\frac{\partial\boldsymbol{\mu}}{\partial[\boldsymbol{\eta}]_k}\right\}. \tag{3}$$

Note that (3) depends on the assumptions on the parameters of the model (equivalently, on the parameter vector $\boldsymbol{\eta}$) via the probability density function $p(\chi|\boldsymbol{\eta})$. The remaining of the section is dedicated to the study of two source models: $i)$ the conditional model for which $\mathbf{CFIM}(\boldsymbol{\eta})$ and $\mathbf{CCRB}(\boldsymbol{\eta})$ will denote the conditional FIM and the conditional CRB w.r.t. the parameter vector $\boldsymbol{\eta}$, respectively; $ii)$ the unconditional model for which $\mathbf{UFIM}(\boldsymbol{\eta})$ and $\mathbf{UCRB}(\boldsymbol{\eta})$ will denote the unconditional FIM and the unconditional CRB w.r.t. the parameter vector $\boldsymbol{\eta}$, respectively. For each case we provide an analytical inversion of the FIM which leads to a non-matrix closed-form expression of the CRB according to the electrical angles. Finally, by using a simple change of variables, we obtain the (non-matrix) expression of CRB according to the physical parameters (bearing and range) for a single source.

A. The Conditional Model

First, let us consider the conditional model. Let us define $\boldsymbol{\psi} = [\psi(1)\cdots\psi(L)]^T$ and $\boldsymbol{\alpha} = [\alpha(1)\cdots\alpha(L)]^T$. The unknown parameter vectors are $\boldsymbol{\xi} = [\omega\ \phi\ \boldsymbol{\psi}^T\ \boldsymbol{\alpha}^T\ \sigma^2]^T$ or $\boldsymbol{\kappa} = [\theta\ r\ \boldsymbol{\psi}^T\ \boldsymbol{\alpha}^T\ \sigma^2]^T$ depending if we are working on the electrical angles or on the physical parameters of interest. First, we derive $\mathbf{CCRB}(\boldsymbol{\xi})$. Second, by using an appropriate change of variables we will deduce $\mathbf{CCRB}(\boldsymbol{\kappa})$. Note that $\boldsymbol{\kappa}$ and $\boldsymbol{\xi}$ are assumed to be deterministic and that their size grows with the number of snapshots. First, let us focus on the derivation of $\mathbf{CCRB}(\boldsymbol{\xi})$. Due to the conditional model assumption we have

$\mathbf{R} = \sigma^2 \mathbf{I}_{NL}$ and $\boldsymbol{\mu} = [s(1)\mathbf{a}^T(\omega,\phi)\cdots s(L)\mathbf{a}^T(\omega,\phi)]^T$. Consequently, by applying (3) one obtains

$$[\mathbf{CFIM}(\boldsymbol{\xi})]_{i,k} = \frac{NL}{\sigma^4}\frac{\partial\sigma^2}{\partial[\boldsymbol{\xi}]_i}\frac{\partial\sigma^2}{\partial[\boldsymbol{\xi}]_k} + \frac{2}{\sigma^2}\Re\left\{\frac{\partial\boldsymbol{\mu}^H}{\partial[\boldsymbol{\xi}]_i}\frac{\partial\boldsymbol{\mu}}{\partial[\boldsymbol{\xi}]_k}\right\}. \quad (4)$$

1) Block-Diagonal Structure of the Fisher Information Matrix: Using (4) and after some tedious, but straightforward, algebraic calculations, one can easily prove the following lemma:

Lemma 1: The structure of $\mathbf{CFIM}(\boldsymbol{\xi})$ for a single near-field source is given by

$$\mathbf{CFIM}(\boldsymbol{\xi}) = \mathrm{bdiag}(\boldsymbol{Q},\boldsymbol{Y}), \quad (5)$$

in which

$$\boldsymbol{Q} = \left[\begin{array}{cc|c} f_{\omega\omega} & f_{\omega\phi} & \mathbf{f}_{\omega\boldsymbol{\psi}} \\ f_{\phi\omega} & f_{\phi\phi} & \mathbf{f}_{\phi\boldsymbol{\psi}} \\ \hline \mathbf{f}_{\boldsymbol{\psi}\omega} & \mathbf{f}_{\boldsymbol{\psi}\phi} & \mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}} \end{array}\right], \quad (6)$$

and $\boldsymbol{Y} = \mathrm{bdiag}((2N/\sigma^2)\mathbf{I}_L, NL/\sigma^4)$, where the conditional SNR is denoted by $\mathrm{C_{SNR}} = \|\boldsymbol{\alpha}\|^2/\sigma^2$, $f_{\omega\omega} = \mathrm{C_{SNR}}LN(N-1)(2N-1)/3$, $f_{\phi\phi} = \mathrm{C_{SNR}}\ LN(N-1)(2N-1)(3N^2-3N-1)/15$, and $f_{\omega\phi} = f_{\phi\omega} = \mathrm{C_{SNR}}\ LN^2(N-1)^2/2$. Furthermore, the $L\times 1$ vectors $\mathbf{f}_{\boldsymbol{\psi}\omega}, (\mathbf{f}_{\omega\boldsymbol{\psi}})^T, \mathbf{f}_{\boldsymbol{\psi}\phi}$ and $(\mathbf{f}_{\phi\boldsymbol{\psi}})^T$ are given by $\mathbf{f}_{\boldsymbol{\psi}\omega} = (\mathbf{f}_{\omega\boldsymbol{\psi}})^T = N(N-1)(\boldsymbol{\alpha}\odot\boldsymbol{\alpha})/\sigma^2$ and $\mathbf{f}_{\boldsymbol{\psi}\phi} = (\mathbf{f}_{\phi\boldsymbol{\psi}})^T = N(N-1)(2N-1)(\boldsymbol{\alpha}\odot\boldsymbol{\alpha})/(3\sigma^2)$. The $L\times L$ matrix $\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}$ is given by $\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}} = 2N\mathrm{diag}(\boldsymbol{\alpha}\odot\boldsymbol{\alpha})/\sigma^2$.

We notice that, thanks to the time-diversity of the source signal, $\mathbf{F}_{\boldsymbol{\alpha}\boldsymbol{\psi}} = (\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\alpha}})^T$ are null matrices. We also note the well-known property that the signal parameters (i.e., $\omega,\phi,\boldsymbol{\psi},\boldsymbol{\alpha}$) are decoupled from the noise variance [19]. The other zero terms are due to the consideration on the real part which appears in (4) applied to purely imaginary quantities and imply that the amplitude of the signal source $\boldsymbol{\alpha}$ is decoupled from the other model signal parameters (i.e., ω,ϕ and $\boldsymbol{\psi}$).

a) Analytical Inversion: Since the size of $\mathbf{CFIM}(\boldsymbol{\xi})$ proposed in (5) is equal to $(2L+3)\times(2L+3)$, it depends on the number of snapshots. A brute-force numerical inversion to obtain $\mathbf{CCRB}(\xi)$ can consequently be a costly operation. Using an appropriate partition of $\mathbf{CFIM}(\boldsymbol{\xi})$ and after writing analytically the expression of the inverse of the Schur complement of the square matrix $\mathbf{F}_{\boldsymbol{\psi}\boldsymbol{\psi}}$ in the upper-left block matrix of $\mathbf{CFIM}(\boldsymbol{\xi})$, we can state the following theorem.

Theorem 1: Non-matrix closed-form expressions of $\mathbf{CCRB}(\boldsymbol{\xi})$ corresponding to the electrical angles, the amplitudes and the shift phases relatively to the model (2) exist *iff* $N \geq 3$ and $\alpha(t)\neq 0\ \forall t = 1\cdots L$. They are expressed as follows:

$$\mathrm{CCRB}(\omega) = \frac{6(2N-1)(8N-11)}{\mathrm{C_{SNR}}\ L(N^2-1)N(N^2-4)}, \quad (7)$$

$$\mathrm{CCRB}(\phi) = \frac{90}{\mathrm{C_{SNR}}\ L(N^2-1)N(N^2-4)}, \quad (8)$$

$$\mathrm{CCRB}(\psi(t)) = \frac{8N^2-12N+4+L\|\boldsymbol{\alpha}\|^2(N^3+3N^2+2N)}{\mathrm{C_{SNR}}\ \alpha^2(t)N^2(N+1)(N+2)},$$

and

$$\mathrm{CCRB}(\alpha(t)) = \frac{\sigma^2}{2N}.$$

Furthermore, the cross terms are given by

$$[\mathbf{CCRB}(\boldsymbol{\xi})]_{1,2} = [\mathbf{CCRB}(\boldsymbol{\xi})]_{2,1} = \frac{-90}{\mathrm{C_{SNR}}\ LN(N^2-4)(N+1)},$$

$$[\mathbf{CCRB}(\boldsymbol{\xi})]_{1,3:3+L} = [\mathbf{CCRB}(\boldsymbol{\xi})]_{3:3+L,1} = \frac{-9(2N-1)}{\mathrm{C_{SNR}}\ LN(N+1)(N+2)}\mathbf{1}_L^T,$$

and

$$[\mathbf{CCRB}(\boldsymbol{\xi})]_{2,3:3+L} = [\mathbf{CCRB}(\boldsymbol{\xi})]_{3:3+L,2} = \frac{15}{\mathrm{C_{SNR}}\ LN(N+1)(N+2)}\mathbf{1}_L^T.$$

Proof: See Appendix A. □

b) Change of Variables: Even if the model (2) is widely used in array signal processing, its CRB relating to $\boldsymbol{\xi}$ does not bring us physical information. Then, it is interesting to analyze the CRB regarding the bearing θ and the range r which are the real physical parameters of the problem. From $\mathbf{CCRB}(\boldsymbol{\xi})$, one can easily obtain $\mathbf{CCRB}(\boldsymbol{\kappa})$ by using a change of variables formula (see [19, p. 45]):

$$\mathbf{CCRB}(\boldsymbol{\kappa}) = \frac{\partial\mathbf{g}(\boldsymbol{\xi})}{\partial\boldsymbol{\xi}^T}\mathbf{CCRB}(\boldsymbol{\xi})\frac{\partial\mathbf{g}^T(\boldsymbol{\xi})}{\partial\boldsymbol{\xi}}$$

where

$$\boldsymbol{\kappa} = \mathbf{g}(\boldsymbol{\xi}) = \left[-\arcsin\left(\tfrac{\omega\lambda}{2\pi d}\right) \quad \tfrac{\pi d^2}{\lambda\phi}\cos^2\left(\arcsin\left(\tfrac{\omega\lambda}{2\pi d}\right)\right) \quad \boldsymbol{\psi}^T \quad \boldsymbol{\alpha}^T\sigma^2\right]^T.$$

Note that the function $\mathbf{g}(\boldsymbol{\xi})$ is well-defined *iff* $\phi\neq 0 \bmod(\pi)$ which implies $\theta\neq\pi/2 \bmod(\pi)$. This condition is intuitive since it corresponds to the ULA ambiguity situation. Then, if $\phi\neq 0\bmod(\pi)$, the Jacobian matrix is given by $\partial\mathbf{g}(\boldsymbol{\xi})/\partial\boldsymbol{\xi}^T = \mathrm{bdiag}(\boldsymbol{A},\boldsymbol{I}_{2L+1})$, where

$$\boldsymbol{A} = \frac{-\lambda}{2\pi d\cos(\theta)}\begin{bmatrix} 1 & 0 \\ -2r\tan(\theta) & \frac{2r^2}{d\cos(\theta)}\end{bmatrix}. \quad (9)$$

Consequently, one obtains the following theorem:

Theorem 2: Non-matrix closed-form expressions of $\mathbf{CCRB}(\boldsymbol{\kappa})$ corresponding to the bearing, the range, the amplitude and the shift phases relatively to the model (2) exist *iff* $N\geq 3$ and $\theta\neq\pi/2\bmod(\pi)$ and $\alpha(t)\neq 0,\ \forall t = 1\cdots L$ and they are given by (10) and (11), shown at the bottom of the page. Furthermore, the cross terms between θ and r are as follows:

$$[\mathbf{CCRB}(\boldsymbol{\kappa})]_{1,2} = [\mathbf{CCRB}(\boldsymbol{\kappa})]_{2,1} = \frac{-3\lambda^2 r}{\mathrm{C_{SNR}}\ L\pi^2 d^3} \times\frac{15r(N-1)+d(8N-11)(2N-1)\sin(\theta)}{N(N^2-1)(N^2-4)\cos^3(\theta)}.$$

$$\mathrm{CCRB}(\theta) = \frac{3\lambda^2}{2\mathrm{C_{SNR}}\ Ld^2\pi^2\cos^2(\theta)}\frac{(8N-11)(2N-1)}{N(N^2-1)(N^2-4)}, \quad (10)$$

$$\mathrm{CCRB}(r) = \frac{6r^2\lambda^2}{\mathrm{C_{SNR}}\ L\pi^2 d^4}\frac{15r^2+30dr(N-1)\sin(\theta)+d^2(8N-11)(2N-1)\sin^2(\theta)}{N(N^2-1)(N^2-4)\cos^4(\theta)}. \quad (11)$$

B. The Unconditional Model

Let us consider now the unconditional model, i.e., when the signals are assumed to be Gaussian (with zero mean and variance σ_s^2) independent of the noise. The unknown parameter vectors are $\boldsymbol{\rho} = [\omega\ \phi\ \sigma_s^2\ \sigma^2]^T$ or $\boldsymbol{\vartheta} = [\theta\ r\ \sigma_s^2\ \sigma^2]^T$ depending if we are working on the electrical angles or on the physical parameters of interest. We first focus on the derivation of $\mathbf{UCRB}(\boldsymbol{\rho})$.

Under the unconditional model assumption, $\boldsymbol{x}(t)|\boldsymbol{\rho} \sim \mathcal{CN}(\mathbf{0},\mathbf{R})\ \forall t = 1,\ldots,L$, where the covariance matrix $\mathbf{R} = \sigma_s^2\mathbf{a}(\omega,\phi)\mathbf{a}^H(\omega,\phi) + \sigma^2\mathbf{I}_N$. Consequently, the FIM in (3) becomes $[\mathbf{UFIM}(\boldsymbol{\rho})]_{i,k} = L\ \mathrm{tr}\{\mathbf{R}^{-1}(\partial\mathbf{R}/\partial[\boldsymbol{\rho}]_i)\mathbf{R}^{-1}(\partial\mathbf{R}/\partial[\boldsymbol{\rho}]_k)\}$. The matrix expression of $[\mathbf{UCRB}(\boldsymbol{\rho})]_{1:2,1:2}$ can be readily established (we omit the proof since it is obtained in the same way as in [13]) according to

$$[\mathbf{UCRB}(\boldsymbol{\rho})]_{1:2,1:2} = \frac{1}{2\mathrm{U_{SNR}}\ \sigma_s^2 L}\left(\Re\left\{\left(\mathbf{D}^H\boldsymbol{\Pi}^{\perp}_{\mathbf{a}(\omega,\phi)}\mathbf{D}\right)\odot\left(\mathbf{J}\otimes\mathbf{a}^H(\omega,\phi)\mathbf{R}^{-1}\mathbf{a}(\omega,\phi)\right)^T\right\}\right)^{-1} \quad (12)$$

where $\mathrm{U_{SNR}} = \sigma_s^2/\sigma^2$ denotes the unconditional SNR, $\mathbf{J} = \mathbf{1}_2\mathbf{1}_2^T$, $\mathbf{D} = [(\partial\mathbf{a}(\omega,\phi)/\partial\omega)(\partial\mathbf{a}(\omega,\phi)/\partial\phi)]$ and $\boldsymbol{\Pi}^{\perp}_{\mathbf{a}(\omega,\phi)} = \mathbf{I}_N - \mathbf{a}(\omega,\phi)(\mathbf{a}^H(\omega,\phi)\mathbf{a}(\omega,\phi))^{-1}\mathbf{a}^H(\omega,\phi)$. In the following we use (12) to derive non-matrix expressions of $\mathbf{UCRB}(\boldsymbol{\rho})$.

1) Analytical Inversion:

Theorem 3: Non-matrix expressions of $\mathbf{UCRB}(\boldsymbol{\rho})$ corresponding to the electrical angles are, well-defined *iff* $N \geq 3$, and are given by

$$\mathrm{UCRB}(\omega) = \left(1+\frac{1}{\mathrm{U_{SNR}}\ N}\right)\frac{6(2N-1)(8N-11)}{\mathrm{U_{SNR}}\ L(N^2-1)N(N^2-4)}, \quad (13)$$

$$\mathrm{UCRB}(\phi) = \left(1+\frac{1}{\mathrm{U_{SNR}}\ N}\right)\frac{90}{\mathrm{U_{SNR}}\ L(N^2-1)N(N^2-4)}. \quad (14)$$

Furthermore, the cross terms are given by

$$[\mathbf{UCRB}(\boldsymbol{\rho})]_{1,2} = [\mathbf{UCRB}(\boldsymbol{\rho})]_{2,1} = -\left(1+\frac{1}{\mathrm{U_{SNR}}\ N}\right)\frac{90}{\mathrm{U_{SNR}}\ LN(N^2-4)(N+1)}.$$

Proof: See Appendix B. □

2) Change of Variables: using the same change of variables formula as for Theorem 2 one can easily prove.

Theorem 4: Non-matrix closed-form expressions of $\mathbf{UCRB}(\boldsymbol{\vartheta})$ corresponding to the range and bearing for a single narrowband near-field source are well-defined *iff* $N \geq 3$ and $\theta \neq \pi/2\mathrm{mod}(\pi)$ and they are expressed in (15) and (16), shown at the bottom of the page. Furthermore, the cross terms between θ and r are given by

$$[\mathbf{UCRB}(\boldsymbol{\vartheta})]_{1,2} = [\mathbf{UCRB}(\boldsymbol{\vartheta})]_{2,1} = -\left(1+\frac{1}{\mathrm{U_{SNR}}\ N}\right)\frac{3\lambda^2 r}{\mathrm{U_{SNR}}\ L\pi^2 d^3} \times \frac{15r(N-1)+d(8N-11)(2N-1)\sin(\theta)}{N(N^2-1)(N^2-4)\cos^3(\theta)}.$$

IV. Analysis of the CRB

The goal of this Section is to validate and analyze the proposed closed-form expressions. The behaviors of the CRB are detailed with respect to physical parameters of the problem.

A. Conditional and Unconditional CRB's Behavior

The scenario used in these simulations is an ULA of $N = 6$ sensors spaced by $d = 0.125$ m. The number of snapshots is equal to $L = 100$ and the location of the source is set as $r = 1.25$ m (which belongs to the Fresnel region according to (1) for $f_0 \in [600, 1200]$ MHz). In Fig. 1, we superimpose the CRBs, obtained from (11) and (16) to the computed CRBs. For these simulations, the signal source is a sample of a complex random Gaussian process with variance $\sigma_s^2 = 10$. The variance of the noise varies from 0.1 to 1.

Moreover, Fig. 1 shows the dependence of the $\mathrm{CCRB}(r)$ and $\mathrm{UCRB}(r)$ w.r.t. the carrier frequency f_0 and suggests that higher is the carrier frequency, lower is the bound. Furthermore, from the closed-form expressions given in (10), (11), (15) and (16), we notice the following.

- UCRB and CCRB are phase-invariant.
- $\mathrm{CCRB}(\theta)$ and $\mathrm{UCRB}(\theta)$ are just bearing-dependent as in the far-field scenario w.r.t. $O(1/\cos^2(\theta))$. It means that the ULA in the near-field case is not isotropic.
- For large N and fixed inter-spacing sensor, $\mathrm{CCRB}(\theta)$ and $\mathrm{UCRB}(\theta)$ in the near-field case tend to the asymptotic CRBs in the far-field case which are given by $(3\lambda^2)/(\mathrm{SNR}2d^2\pi^2\cos^2(\theta)N^3)$. This is consistent with the intuition since, due to the Fresnel constraint, large N implies large range, which corresponds to the far-field scenario.
- $\mathrm{CCRB}(r)$ and $\mathrm{UCRB}(r)$ are bearing-dependent and range-dependent. For r proportional to d, the dependence w.r.t. the range is $O(r^2)$, meaning that nearer is the source better is the range estimation (keeping in mind the Fresnel constraints).
- The dependence of the range w.r.t. the bearing is $O(1/\cos^4(\theta))$. For θ close to $\pi/2$ (i.e., close to the ambiguity situation), we observe that $\mathrm{CCRB}(r)$ and $\mathrm{UCRB}(r)$ go to infinity but faster than $\mathrm{CCRB}(\theta)$ and $\mathrm{UCRB}(\theta)$, respectively.
- For a sufficient number of sensors, $\mathrm{CCRB}(\theta), \mathrm{UCRB}(\theta)$, $\mathrm{CCRB}(r)$ and the $\mathrm{UCRB}(r)$ are $O(1/N^3)$.
- For λ proportional to $d, \mathrm{CCRB}(\theta)$ and $\mathrm{UCRB}(\theta)$ are independent of the carrier frequency f_0. This is not the case for $\mathrm{CCRB}(r)$ and $\mathrm{UCRB}(r)$. Furthermore, note that higher is the carrier frequency, better is the estimation of the range (cf. Fig. 1).
- Note that the expressions of $[\mathbf{CCRB}(\boldsymbol{\kappa})]_{1,2}, [\mathbf{CCRB}(\boldsymbol{\kappa})]_{2,1}$, $[\mathbf{UCRB}(\boldsymbol{\vartheta})]_{1,2}$ and $[\mathbf{UCRB}(\boldsymbol{\vartheta})]_{2,1}$ show that the physical parameters of interest are strongly coupled since $[\mathbf{CCRB}(\boldsymbol{\kappa})]_{1,2}$ and $[\mathbf{UCRB}(\boldsymbol{\vartheta})]_{1,2}$ are $O(1/N^3)$ as $\mathrm{CCRB}(\theta), \mathrm{CCRB}(r), \mathrm{UCRB}(\theta)$ and $\mathrm{UCRB}(r)$.
- Finally, since $\mathrm{UCRB}(\omega)$ is $O(1/N^3)$ and $\mathrm{UCRB}(\phi)$ is $O(1/N^5)$, thus, for a sufficient number of sensors the estimation of the so-called second electrical angle ϕ is more accurate than estimating the first electrical angle ω.

$$\mathrm{UCRB}(\theta) = \left(1+\frac{1}{\mathrm{U_{SNR}}\ N}\right)\frac{3\lambda^2}{2\mathrm{U_{SNR}}\ Ld^2\pi^2\cos^2(\theta)}\frac{(8N-11)(2N-1)}{N(N^2-1)(N^2-4)}, \quad (15)$$

$$\mathrm{UCRB}(r) = \left(1+\frac{1}{\mathrm{U_{SNR}}\ N}\right)\frac{6r^2\lambda^2}{\mathrm{U_{SNR}}\ L\pi^2 d^4}\frac{15r^2+30dr(N-1)\sin(\theta)+d^2(8N-11)(2N-1)\sin^2(\theta)}{N^2(N^2-1)(N^2-4)\cos^4(\theta)}. \quad (16)$$

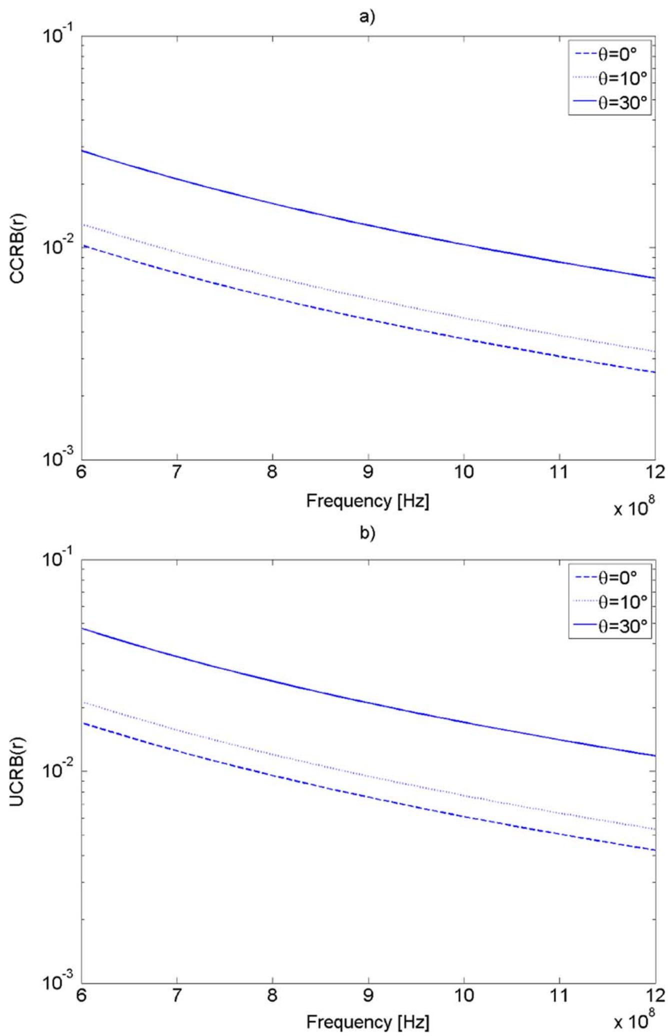

Fig. 1. $\mathrm{CRB}(r)$ versus f_0 for $\sigma^2 = 0.5$ and different values of $\theta = 10°, 30°, 50°$: (a) $\mathrm{CCRB}(r)$ and (b) $\mathrm{UCRB}(r)$.

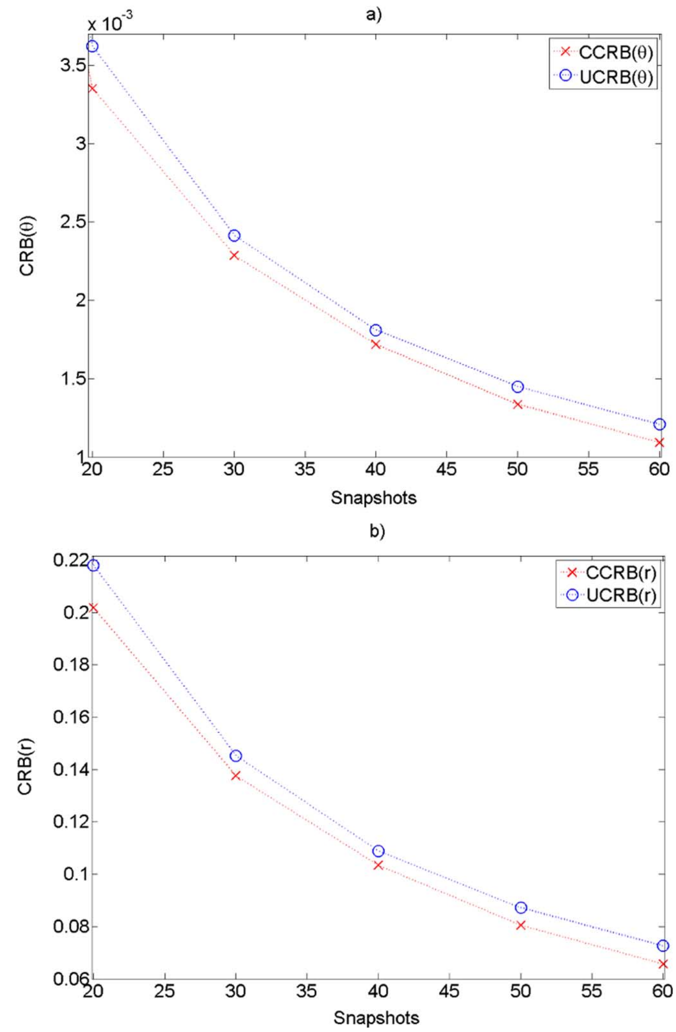

Fig. 2. CRBs versus the number of snaphots for $N = 10$: (a) $\mathrm{CCRB}(\theta)$ and $\mathrm{UCRB}(\theta)$, (b) $\mathrm{CCRB}(r)$ and $\mathrm{UCRB}(r)$.

B. Analytical and Numerical Comparison Between the CCRB and the UCRB

Since the conditional model does not make any assumptions on the source, we can chose the phase and the amplitude of the source as samples of a random process. In this case, we can study an analytical and numerical comparison between the conditional and the unconditional CRB. Furthermore, we assume that the two physical quantities $\mathrm{C_{SNR}}$ and $\mathrm{U_{SNR}}$ are equals to the same quantity denoted by SNR.

Corollary 1: From (7) and (13), one obtains the following: $\mathrm{UCRB}(\omega)/\mathrm{CCRB}(\omega) = 1 + (1/(\mathrm{SNR}\ N))$, and $\mathrm{UCRB}(\phi)/\mathrm{CCRB}(\phi) = 1 + (1/(\mathrm{SNR}\ N))$. In the same way, from (10) and (15), one obtains the following: $\mathrm{UCRB}(\theta)/\mathrm{CCRB}(\theta) = 1 + (1/(\mathrm{SNR}\ N))$, and $\mathrm{UCRB}(r)/\mathrm{CCRB}(r) = 1 + (1/(\mathrm{SNR}\ N))$, i.e., $\mathrm{UCRB}(\omega) \geq \mathrm{CCRB}(\omega), \mathrm{UCRB}(\phi) \geq \mathrm{CCRB}(\phi)$ and $\mathrm{UCRB}(\theta) \geq \mathrm{CCRB}(\theta)$, $\mathrm{UCRB}(r) \geq \mathrm{CCRB}(r)$ (cf. Fig. 2). Note that, a similar result has been shown in the far-field case in [12].

Furthermore,

- for a fixed N: $\mathrm{CCRB}(\theta) \overset{\mathrm{SNR}\to\infty}{\to} \mathrm{UCRB}(\theta)$, and $\mathrm{CCRB}(r) \overset{\mathrm{SNR}\to\infty}{\to} \mathrm{UCRB}(r)$.
- for a fixed SNR: $\mathrm{CCRB}(\theta) \overset{N\to\infty}{\to} \mathrm{UCRB}(\theta)$ and $\mathrm{CCRB}(r) \overset{N\to\infty}{\to} \mathrm{UCRB}(r)$.
- and finally, for $1/(\mathrm{SNR}\ N) \ll 1$: $\mathrm{CCRB}(\theta) \approx \mathrm{UCRB}(\theta)$ and $\mathrm{CCRB}(r) \approx \mathrm{UCRB}(r)$.

V. Conclusion

In this correspondence, the conditional and the unconditional Cramér–Rao bounds are derived in a closed-form expressions for a single near-field time-varying narrowband source in terms of range and bearing. These expressions are given in non-matrix forms which are important in order to avoid a costly Fisher information matrix numerical inversion. Moreover these expressions provide useful information concerning the behavior of the bounds. In this way, the proposed expressions have been analyzed with respect to the physical parameters of the problem. In particular, we provided an explicit link between the conditional and the unconditional CRB and we shown that higher is the carrier frequency and/or closer is the source from the array, better is the estimation of the range.

Appendix A

In this Appendix, we highlight the major steps leading to Theorem 1. From (5) one has,

$$\begin{aligned}\det\{\mathbf{CFIM}(\boldsymbol{\xi})\} &= \det\{\boldsymbol{Q}\}\det\{\boldsymbol{Y}\} \\ &= \det\left\{\boldsymbol{\Lambda}_{\mathbf{F}_{\boldsymbol{\psi\psi}}}\right\}\det\{\mathbf{F}_{\boldsymbol{\psi\psi}}\}\det\{\boldsymbol{Y}\} \qquad (17)\end{aligned}$$

$$\mathbf{D}^H \mathbf{\Pi}_{\mathbf{a}}^{\perp} \mathbf{D} = \mathbf{D}^H \mathbf{D} - \frac{1}{N}(\mathbf{a}^H \mathbf{D})^H (\mathbf{a}^H \mathbf{D}) = \begin{bmatrix} \sum_{n=0}^{N-1} n^2 - \frac{1}{N}\left(\sum_{n=0}^{N-1} n\right)^2 & \sum_{n=0}^{N-1} n^3 - \frac{1}{N}\left(\sum_{n=0}^{N-1} n\right)\left(\sum_{n=0}^{N-1} n^2\right) \\ \sum_{n=0}^{N-1} n^3 - \frac{1}{N}\left(\sum_{n=0}^{N-1} n\right)\left(\sum_{n=0}^{N-1} n^2\right) & \sum_{n=0}^{N-1} n^4 - \frac{1}{N}\left(\sum_{n=0}^{N-1} n^2\right)^2 \end{bmatrix}.$$

where $\mathbf{\Lambda}_{\mathbf{F}_{\psi\psi}}$ denotes the Schur complement w.r.t. the matrix $\mathbf{F}_{\psi\psi}$. Assuming that $\alpha(t) \neq 0, \ \forall t = 1 \cdots L, \mathbf{F}_{\psi\psi}$ is invertible and the Schur complement is expressed as follows:

$$\mathbf{\Lambda}_{\mathbf{F}_{\psi\psi}} = \begin{bmatrix} f_{\omega\omega} & f_{\omega\phi} \\ f_{\phi\omega} & f_{\phi\phi} \end{bmatrix} - \begin{bmatrix} \mathbf{f}_{\omega\psi} \\ \mathbf{f}_{\phi\psi} \end{bmatrix} \mathbf{F}_{\psi\psi}^{-1} [\mathbf{f}_{\psi\omega} \quad \mathbf{f}_{\psi\phi}] = \boldsymbol{B} - \frac{\sigma^2}{2N} \boldsymbol{W} \mathrm{diag}(\boldsymbol{\alpha} \odot \boldsymbol{\alpha})^{-1} \boldsymbol{W}^T \tag{18}$$

where

$$\boldsymbol{B} = \mathrm{C}_{\mathrm{SNR}} \ LN(N-1) \begin{bmatrix} \frac{(2N-1)}{3} & \frac{N(N-1)}{2} \\ \frac{N(N-1)}{2} & \frac{(2N-1)(3N^2-3N-1)}{15} \end{bmatrix},$$

and

$$\boldsymbol{W} = \frac{1}{\sigma^2} \begin{bmatrix} N(N-1) \\ \frac{N(N-1)(2N-1)}{3} \end{bmatrix} \otimes (\boldsymbol{\alpha} \odot \boldsymbol{\alpha})^T.$$

Thus, by replacing (18) in (17) one obtains

$$\det\{\mathbf{CFIM}(\boldsymbol{\xi})\} = \frac{1}{540}\left(\frac{2N}{\sigma^2}\right)^{2L} N^2(N^2-4)(N-1)^2(N+1)^2 \|\boldsymbol{\alpha}\|^2 \prod_{t=1}^{L} \alpha^2(t).$$

Consequently, $\det\{\mathbf{CFIM}(\boldsymbol{\xi})\} \neq 0$ *iff* $N \geq 3$ and $\alpha(t) \neq 0 \ \forall t = 1 \ldots L$. Assuming $N \geq 3$ and $\alpha(t) \neq 0 \ \forall t = 1 \ldots L$, one has

$$\mathbf{CFIM}^{-1}(\boldsymbol{\xi}) = \mathrm{bdiag}(\boldsymbol{Q}^{-1}, \boldsymbol{Y}^{-1})$$

where

$$\boldsymbol{Y}^{-1} = \mathrm{bdiag}\left(\frac{\sigma^2}{2N}\mathbf{I}_L, \frac{\sigma^4}{NL}\right).$$

In order to derive $\boldsymbol{Q}^{-1}$, we use the Schur complement $\mathbf{\Lambda}_{\mathbf{F}_{\psi\psi}}$ given in (18). Thus,

$$[\mathbf{CCRB}(\boldsymbol{\xi})]_{1:2,1:2} = \mathbf{\Lambda}_{\mathbf{F}_{\psi\psi}}^{-1}.$$

Since the Schur complement $\mathbf{\Lambda}_{\mathbf{F}_{\psi\psi}}$ is a 2×2 matrix, its inverse is easily derivable and leads to (7), (8). The other terms are directly derived from the following calculation, where

$$\mathbf{CCRB}(\boldsymbol{\psi}) = \frac{\sigma^2}{2N}\mathrm{diag}(\alpha \odot \boldsymbol{\alpha})^{-1} \times \left(\mathbf{I}_L + \frac{\sigma^2}{2N}\boldsymbol{W}^T \mathbf{\Lambda}_{\mathbf{F}_{\psi\psi}}^{-1} \boldsymbol{W} \mathrm{diag}(\alpha \odot \boldsymbol{\alpha})^{-1}\right),$$

$$[\mathbf{CCRB}(\boldsymbol{\xi})]_{1:2,3:L+2} = ([\mathbf{CCRB}(\boldsymbol{\xi})]_{3:L+2,1:2})^H = \frac{-\sigma^2}{2N^2}\mathbf{\Lambda}_{\mathbf{F}_{\psi\psi}}^{-1} \boldsymbol{W} \mathrm{diag}(\boldsymbol{\alpha} \odot \boldsymbol{\alpha})^{-1}.$$

Appendix B

In this Appendix, the dependence on (ω, ϕ) of $\boldsymbol{a}(\omega, \phi)$ is omitted for sake of simplicity. Applying the matrix inversion lemma [18] to $\mathbf{R}$, one obtains

$$\mathbf{R}^{-1} = \frac{1}{\sigma_s^2}\left(\mathbf{a}\mathbf{a}^H + \frac{1}{\mathrm{U}_{\mathrm{SNR}}}\mathbf{I}_N\right)^{-1} = \frac{\mathrm{U}_{\mathrm{SNR}}}{\sigma_s^2}\left(\mathbf{I}_N - \mathrm{U}_{\mathrm{SNR}}\frac{\mathbf{a}\mathbf{a}^H}{1+\mathrm{U}_{\mathrm{SNR}} N}\right).$$

Thus, using the above equation one has

$$\mathbf{J} \otimes \left(\sigma_s^2 \mathbf{a}^H \mathbf{R}^{-1} \mathbf{a} \sigma_s^2\right)^T = \sigma_s^2 \mathrm{U}_{\mathrm{SNR}}\left(N - \frac{\mathrm{U}_{\mathrm{SNR}} \ N^2}{1+\mathrm{U}_{\mathrm{SNR}} \ N}\right)\mathbf{J}. \tag{19}$$

On the other hand, the derivation of $\boldsymbol{a}$ w.r.t. ω and ϕ leads to

$$[\mathbf{D}]_{i,k} = j((i-1)\delta(k-1) + (i-1)^2\delta(k-2))e^{j(\omega n + \phi n^2)} \quad \forall i = 1 \cdots N \text{ and } \forall k = 1, 2.$$

Consequently [see the equation shown at the top of the page]. Thus, using the above expression and (19) and after some simplifications, we obtain

$$\Re\left\{\sigma_s^4\left(\mathbf{D}^H \mathbf{\Pi}_{\mathbf{a}}^{\perp} \mathbf{D}\right) \odot (\mathbf{J} \otimes \mathbf{a}^H \mathbf{R}^{-1} \mathbf{a})^T\right\} = \frac{\mathrm{U}_{\mathrm{SNR}}^2 \ LN(N^2-4)}{6\mathrm{U}_{\mathrm{SNR}} \ N+6}\begin{bmatrix} \frac{15}{(N^2-1)} & \frac{15}{(N+1)} \\ \frac{15}{(N+1)} & \frac{(2N-1)(8N-11)}{(N^2-1)} \end{bmatrix} \tag{20}$$

which leads to

$$\det\left\{\sigma_s^4 \Re\left\{\left(\mathbf{D}^H \mathbf{\Pi}_{\mathbf{a}}^{\perp} \mathbf{D}\right) \odot (\mathbf{J} \otimes \mathbf{a}^H \mathbf{R}^{-1} \mathbf{a})^T\right\}\right\} = \frac{1}{540}\mathrm{U}_{\mathrm{SNR}}\left(1 + \frac{\mathrm{U}_{\mathrm{SNR}}}{N}\right) N^2(N^2-4)(N-1)^2(N+1)^2.$$

Consequently,

$$\det\left\{\sigma_s^4 \Re\left\{\left(\mathbf{D}^H \mathbf{\Pi}_{\mathbf{a}}^{\perp} \mathbf{D}\right) \odot (\mathbf{J} \otimes \mathbf{a}^H \mathbf{R}^{-1} \mathbf{a})^T\right\}\right\} \neq 0 \Leftrightarrow N \geq 3.$$

Then, assuming that $N \geq 3$ and replacing (20) in (12), we obtain

$$[\mathbf{UCRB}(\boldsymbol{\rho})]_{1:2,1:2} = \left(1 + \frac{1}{\mathrm{U}_{\mathrm{SNR}} \ N}\right)\frac{6}{\mathrm{U}_{\mathrm{SNR}} \ L(N^2-4)} \times \begin{bmatrix} \frac{(2N-1)(8N-11)}{(N^2-1)} & -\frac{15}{(N+1)} \\ -\frac{15}{(N+1)} & \frac{15}{(N^2-1)} \end{bmatrix}.$$

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Optimal Relay Function in the Low-Power Regime for Distributed Estimation Over a MAC

Marco Guerriero, Stefano Marano, Vincenzo Matta, and Peter Willett, *Fellow, IEEE*

***Abstract*—A random parameter is estimated by a distributed network of sensors that communicate over a common multiple-access channel (MAC). A MAC implies an additive fusion rule, and the goal here is to design a power-constrained forwarding strategy and fusion center post-processing. To get an explicit solution we appeal to asymptotics, meaning that we design the *locally* optimal scheme for the limiting case that the received power goes to zero.**

***Index Terms*— Distributed estimation, MAC, relay.**

I. Introduction and Background

Reliable data delivery from several remote sensors sharing a common transmission medium is possible, and can be realized by employing source and channel coding strategies borrowed from (or extending) classical point-to-point results. This approach decouples the source and channel coding stages; however, it is known that this separation is not necessarily optimal [1] when the sources are correlated and/or one's goal is not direct recovery of the observations.

A basic lesson from [2] is that there exist cases in which a simple amplify-and-forward strategy outperforms the best separate scheme by orders of magnitude. This happens, for instance, for Gaussian estimation problems over a Gaussian MAC [3], and it is due to the perfect match between the (additive) nature of the optimal MMSE estimator and the (additive) channel structure. In this work, we depart from the source/channel Gaussian model and the corresponding amplify-and-forward solution, allowing the local encoders to apply a nonlinear transformation to arbitrarily distributed observations (see Fig. 1). We limit ourselves to the ideal MAC—not necessarily a Gaussian one—that requires perfect synchronization of the transmissions, both in time and phase, at the local sensors, and we operate under a relayed power constraint.

Joint consideration of the estimation problem and the noisy MAC is in [4] and [5], in which in the asymptote of an increasingly large number of sensors an optimal communication/estimation scheme (called type-based multiple access, or TBMA) was proposed for quantized observations. A likelihood-based multiple access (LBMA) scheme, suitable for continuous observations, was discussed in [6], and yields asymptotically efficient estimation over a waveform channel.

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M. Guerriero and P. Willett are with the Department of Electrical and Computer Engineering, University of Connecticut, U-2157, Storrs, CT 06269 USA (e-mail: marco.guerriero@engr.uconn.edu; willett@engr.uconn.edu; p.willett@ieee.org).

S. Marano and V. Matta are with the Department of Information and Electrical Engineering (DIIIE), University of Salerno, 84084 Fisciano (SA), Italy (e-mail: marano@unisa.it; vmatta@unisa.it).

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