Exercice 1 :

1) $f_X(x) \geqslant 0$ car pour $x \in [0,a]$, $1-\frac{x}{a} \geqslant 0$ (et $a>0$)

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = \frac{2}{a}\int_0^a 1-\frac{x}{a}\,dx = \frac{2}{a}\left([x]_0^a - \frac{1}{a}\left[\frac{x^2}{2}\right]_0^a \right) = \frac{2}{a}\left(a - \frac{1}{a}\frac{a^2}{2}\right) = 1$$

2) $X \in L^1$ (et L^2) car la fct $x f_X(x)$ est bornée sur un support borné

$$E(X) = \frac{2}{a}\int_0^a x\left(1-\frac{x}{a}\right)dx = \frac{2}{a}\left(\left[\frac{x^2}{2}\right]_0^a - \frac{1}{a}\left[\frac{x^3}{3}\right]_0^a\right) = \frac{2}{a}\left(\frac{a^2}{2} - \frac{a^2}{3}\right) = \boxed{a/3}$$

3) $$\mathrm{Var}(X) = \frac{2}{a}\int_0^a x^2\left(1-\frac{x}{a}\right)dx - \left(\frac{a}{3}\right)^2 = a^2/6 - a^2/9 = \boxed{a^2/18}$$

4) Si $x \leqslant 0 \Rightarrow F_X(x) = 0$, si $x \geqslant a \Rightarrow F_X(x) = 1$ et si $x \in [0,a]$

$$F_X(x) = \int_0^x \frac{2}{a}\left(1-\frac{u}{a}\right)dx = \frac{2}{a}\left(\left[u\right]_0^x - \frac{1}{a}\left[\frac{u^2}{2}\right]_0^x\right) = \frac{(2a-x)}{a^2}x$$

5) $$P(X \leqslant m) = \frac{(2a-m)}{a^2}m = \frac{1}{2} \Leftrightarrow m^2 - 2am + \frac{a^2}{2} = 0 \; ; \; \Delta = 4a^2 - 4\frac{a^2}{2} = 2a^2 > 0$$

$$\Rightarrow m = \frac{2a \pm \sqrt{2a^2}}{2} = a \pm \frac{a}{\sqrt{2}} \text{ or } m \in [0,a] \Rightarrow \boxed{m = a\left(1-\frac{1}{\sqrt{2}}\right)}$$

6) $$\phi_X(u) = E\left(e^{iuX}\right) = \int_0^a \frac{2}{a}\left(1-\frac{x}{a}\right)e^{iux}dx = \frac{2}{a}\left(\int_0^a e^{iux} - \frac{1}{a}\int_0^a x e^{iux}\,dx\right)$$

$$= \frac{2}{a}\left(\frac{e^{iua}-1}{iu} - \frac{1}{a}\left(\frac{a e^{iua}}{iu} - \frac{1}{iu}\left[\frac{e^{iux}}{iu}\right]_0^a\right)\right)$$

$$= \frac{2}{aiu}\left(\frac{e^{iua}-1}{aiu} - 1\right) = \boxed{\frac{2}{a^2u^2}\left(aiu + 1 - e^{iua}\right)}$$

7) $Y = X^2+1 \Rightarrow$

$$\int_0^a h(x) f_X(x)\,dx = \int_0^a h(x)\frac{2}{a}\left(1-\frac{x}{a}\right)dx = \int_1^{a^2+1} h(\sqrt{y-1})\frac{2}{a}\left(1-\frac{\sqrt{y-1}}{a}\right) \times \frac{1}{2\sqrt{y-1}}\,dy$$

$x = \sqrt{y-1}$

$\frac{dx}{dy} = \frac{1}{2\sqrt{y-1}}$

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} \frac{1}{a}\left(\frac{1}{\sqrt{y-1}} - \frac{1}{a}\right) & \text{si } y \in [1, a^2+1] \\ 0 \text{ sinon} \end{cases}}$$

Exercice 2 :

1) $$P(A_i) = P(R_c \mid V_R)P(V_R) + P(R_c \mid \overline{V_R})P(\overline{V_R}) = 1 \times p + \frac{1}{4}(1-p) = \boxed{\frac{3}{4}p + \frac{1}{4}}$$

2) $S \sim \mathrm{Bin}\left(40, \frac{3}{4}p + \frac{1}{4}\right)$ (question indépendantes)

3) On veut $E(S) = 36$. Or $E(S) = 40\left(\frac{3}{4}p + \frac{1}{4}\right) = 36 \Leftrightarrow \boxed{p = 13/15}$

4) $P(S=40) = \frac{40!}{40!\,0!}\left(\frac{3}{4}\cdot\frac{13}{15}+\frac{1}{4}\right)^{40}(\ .\ )^{0} = \boxed{1,47\%}$

Exercice 4 :

1) $X \sim \mathcal{N}(0,4)$ et $Y \sim \mathcal{N}(0,1) \Rightarrow \mathbb{E}(2X+Y) = 2\,\mathbb{E}(X)+\mathbb{E}(Y) = \boxed{0}$

et 2) $\mathbb{E}(X-3Y) = \mathbb{E}(X) - 3\,\mathbb{E}(Y) = \boxed{0}$

$Var(2X+Y) = \mathbb{E}((2X+Y)^2) = 4\,\mathbb{E}(X^2) + \mathbb{E}(Y^2) + 4\,\mathbb{E}(XY) = 16+1+4\,\mathbb{E}(XY)$

Or $\mathbb{E}((2X+Y)(X-3Y)) = 0 \Leftrightarrow \mathbb{E}(2X^2 - 6XY + XY - 3Y^2) = 0 \Leftrightarrow 2\,\mathbb{E}(X^2) - 5\,\mathbb{E}(XY) - 3\,\mathbb{E}(Y^2) = 0$

$\Leftrightarrow \boxed{\mathbb{E}(XY) = 1}$

D'où $Var(2X+Y) = 17+4 = \boxed{21}$ et $Var(X-3Y) = \mathbb{E}((X-3Y)^2)$
$= \mathbb{E}(X^2) - 6\,\mathbb{E}(XY) + 9\,\mathbb{E}(Y^2)$
$= 4 - 6 + 9 = \boxed{7}$

3) $\Gamma_{XY} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$; X et Y pas-dépendantes décorrélé $\Leftrightarrow$

4) $f_{X,Y}(x,y) = \frac{1}{2\pi|\Gamma_{XY}|^{1/2}} \exp\left\{-\frac{1}{2}(x\ y)\,\Gamma_{XY}^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right\}$ or $|\Gamma_{XY}| = 3$ et $\Gamma_{XY}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$

$= \frac{1}{2\sqrt{3}\pi} \exp\left\{-\frac{1}{6}(x\ y)\begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right\} = \frac{1}{2\sqrt{3}\pi}\exp\left\{-\frac{1}{6}(x^2 - 2xy + 4y^2)\right\}$

Exercice 5 :

1) $P(E_5 \cup E_6) = 1 - P(E_4) = \boxed{70\%}$ 2) $P(E_4|PE) = P(PE|E_4)\frac{P(E_4)}{P(PE)} = \frac{0,3\times 0,3}{0,34} = \boxed{26,5\%}$

3) $P(PE) = P(PE|E_4)P(E_4) + P(PE|E_5)P(E_5) + P(PE|E_6)P(E_6)$
$= 0,3\times 0,3 + 0,5\,P(E_5) + 0,4\,P(E_6) = 0,34 \Rightarrow \frac{8}{5}P(E_6) + P(E_5) = 1$

4) $\begin{cases} P(E_5)+P(E_6) = 0,7 \\ P(E_5) + \frac{8}{5}P(E_6) = 1 \end{cases} \Leftrightarrow \begin{cases} 1 - \frac{8}{5}P(E_6) + P(E_6) = 0,7 \\ P(E_5)+P(E_6) = 0,7 \end{cases} \Leftrightarrow \begin{cases} \boxed{P(E_6) = \frac{1}{2}} \\ P(E_5) = 0,7 - 0, \end{cases}$

5) $P(E_6 | PE \cap \overline{E_5}) = \frac{P(PE \cap \overline{E_5} \cap E_6)}{P(PE \cap \overline{E_5})} = \frac{P(PE \cap E_6)}{P(PE \cap \overline{E_5})} = \frac{P(PE|E_6)P(E_6)}{P(PE \cap (E_4 \cup E_6))}$ $\boxed{= 0,2}$

$= \frac{P(PE|E_6)P(E_6)}{P(PE|E_4)P(E_4) + P(PE|E_6)P(E_6)} = \frac{0,4\times 0,5}{0,3\times 0,3 + 0,4\times 0,5}$ disjoint $= \boxed{69\%}$

Exercice 3 :

1) $X_1 \sim geom\{p\} \Rightarrow$ $X_1 \in \{1,2,\dots\}$
$P(X_1 = k) = p(1-p)^{k-1}$ $\forall k \in \mathbb{N}^*$

$\mathbb{E}(X_1) = 1/p$ et $Var(X_1) = \frac{1-p}{p^2}$

2) $P(X_3 > m) = \sum_{k=m+1}^{+\infty} P(X_3 = k) = p\sum_{k=m+1}^{+\infty}(1-p)^{k-1} = p\sum_{k'=0}^{+\infty}(1-p)^{k'+m}$ $k' = k-(m+1)$

$= p(1-p)^m \frac{1}{1-(1-p)} = \boxed{(1-p)^m}$

3] $\Delta \in \mathbb{N}$

4] $P(\Delta=0) = P(X_1=1 \cap X_2=1 \cup X_1=2 \cap X_2=2 \cup \dots)$

$$= \sum_{k=1}^{+\infty} P(X_1=k \cap X_2=k) = \sum_{k=1}^{+\infty} P(X_1=k)P(X_2=k)$$

$$= p^2 \sum_{k=1}^{+\infty} \left((1-p)^2\right)^{k-1} = \frac{p^2}{1-(1-p)^2} = \boxed{\frac{p}{2-p}}$$

5] $P(X_1-X_2=m) = P(X_1=1+m \cap X_2=1 \cup X_1=2+m \cap X_2=2 \cup \dots)$

$$= \sum_{k=1}^{+\infty} P(X_1=k+m \cap X_2=k) = \sum_{k=1}^{+\infty} P(X_1=k+m)P(X_2=k)$$

disjoit — idepedace

$\Rightarrow P(\Delta=m) = P(X_1-X_2=m \cup X_2-X_1=m) = 2P(X_1-X_2=m)$

$$= 2\sum_{k=1}^{+\infty} p^2(1-p)^{k+m-1}(1-p)^{k-1}$$ disjoit — par symétrie

$$= 2p^2(1-p)^m \sum_{k=1}^{+\infty} \left((1-p)^2\right)^{k-1} = \frac{2p^2(1-p)^m}{1-(1-p)^2} = \boxed{\frac{2(1-p)^m p}{2-p}}$$

6] $E(\Delta) = \sum_{m=0}^{+\infty} m\,P(\Delta=m) = \sum_{m=1}^{+\infty} m \frac{2(1-p)^m p}{2-p} = \frac{2p}{2-p} \sum_{m=1}^{+\infty} \underbrace{m(1-p)^m}$

$\Delta \in L^1$ si $\lim_{m\to+\infty} \left|\frac{u_{m+1}}{u_m}\right| < 1$. Or $\lim_{m\to+\infty} \left|\frac{u_{m+1}}{u_m}\right| = \lim_{m\to+\infty} \frac{(m+1)(1-p)^{m+1}}{m(1-p)^m}$ (u_m) $= 1-p < 1$ puisque $0<p<1$

$$\Rightarrow E(\Delta) = \frac{2p}{2-p}(1-p)\sum_{m=1}^{+\infty} m(1-p)^{m-1} = \boxed{\frac{2(1-p)}{p(2-p)}}$$

7] X_1 et X_2 on $\hat{m}$ distribution ($\Rightarrow \hat{m}$ E et Var)

$$\Rightarrow E\left((X_1-X_2)^2\right) = E(X_1^2+X_2^2-2X_1X_2) = E(X_1^2) + \underbrace{E(X_2^2)}_{=E(X_1^2)} - 2E(X_1)^2$$

$\Rightarrow \Delta \in L^2$ car $E(|X_1-X_2|^2) = E((X_1-X_2)^2) = 2\left(E(X_1^2)-(E(X_1))^2\right)$

et $Var(X_1)$ existe ($X_1 \in L^2$)

8] $P(A) = P(X_3>\Delta) = P(X_3>0 \cap \Delta=0 \cup X_3>1 \cap \Delta=1 \cup \dots)$

$$= \sum_{m=0}^{+\infty} P(X_3>m \mid \cancel{\Delta=m})P(\Delta=m)$$

disjoit — idepedat

9] $P(A) = \sum_{m=1}^{+\infty} (1-p)^m \frac{2(1-p)^m p}{2-p} + \frac{p}{2-p}\underbrace{P(X_3>0)}_{=1} = \frac{p}{2-p} + \frac{2p}{2-p}\sum_{m=1}^{+\infty}\left[(1-p)^2\right]^m$

$$= \frac{p}{2-p} + \frac{2p}{2-p}\left(\frac{1}{1-(1-p)^2} - 1\right) = \frac{p}{2-p} + \frac{2p}{2-p}\left(\frac{1}{p(2-p)} - 1\right)$$

$$= \boxed{\frac{p^2-2p-2}{(2-p)^2} \text{ ou } \frac{1+(1-p)^2}{(2-p)^2}}$$