Partiel Probabilités UE 332 2024

Exercice 1 : C_E = "classe éco" ; S_c = "séjours court"

1) $P(C_E) = \frac{220}{275} = 0,8$ 2) $P(S_c | \overline{C_E}) = 1 - P(\overline{S_c} | \overline{C_E}) = 0,3$

3) $P(C_E \cap \overline{S_c}) = P(\overline{S_c} | C_E) P(C_E) = 0,35 \times 0,8 = 0,28$

4) $P(C_E | \overline{S_c}) = \frac{P(C_E \cap \overline{S_c})}{P(\overline{S_c})} = \frac{0,28}{P(\overline{S_c}|C_E) P(C_E) + P(\overline{S_c} | \overline{C_E}) P(\overline{C_E})}$

$= \frac{0,28}{0,42} = 0,667$

Exercice 2 :

1) $X \sim Geom(\frac{1}{2})$ 2) $R = \frac{1}{X} \Rightarrow \underbrace{\mathbb{E}(R)}_{= \mathbb{E}(\frac{1}{X})} = \sum_{k=1}^{+\infty} \frac{1}{k} \frac{1}{2} \underbrace{(1 - \frac{1}{2})}_{= 1/2}^{k-1}$

$= \sum_{k=1}^{+\infty} \frac{2^{-k}}{k} = -\ln(1 - \frac{1}{2})$

$= \ln 2$

Exercice 3 :

1) $P(\text{"serveur en panne"}) = P(X \geq 2)$ avec $X = A_1 + A_2 + A_3$

Puisque $A_i \sim Ber(p)$ + indépendance $\Rightarrow X \sim Bin(3, p)$

$\Rightarrow P(X \geq 2) = P(X = 2 \cup X = 3) = P(X=2) + P(X=3) = 3p^2(1-p) + p^3$

$= p^2(3(1-p) + p) = p^2(-2p+3)$

2) $P(\text{"serveur en panne"}) = P(Y \geq 2)$ avec $Y \sim Bin(4, q)$

$= P(Y=2 \cup Y=3 \cup Y=4) = P(Y=2) + P(Y=3) + P(Y=4)$

$= 6q^2(1-q)^2 + 4q^3(1-q) + q^4$

$= q^2(3q^2 - 8q + 6)$

3) si $p = q$ alors $\alpha - \beta = \underbrace{p^2}_{\geq 0}(-3p^2 + 6p - 3)$

$\hookrightarrow \Delta = 36 - 4 \times 3 \times 3 = 0$

$\Rightarrow p = 1 \Leftrightarrow -3(p-1)^2$

$\Rightarrow \alpha - \beta \leq 0 \Rightarrow$ le cas 1 est préférable

Exercice 4 :

1) $X_i = \{1, 2, 3, \ldots, 365\}$ avec $P(X_i = j) = \frac{1}{365}$

2) $P(X_i = X_{i'}) = P(X_i = 1 \cap X_{i'} = 1 \cup X_i = 2 \cap X_{i'} = 2 \cup \ldots \cup X_i = 365 \cap X_{i'} = 365)$

$\underset{\text{Disjoint + Indep}}{=} \sum_{k=1}^{365} P(X_i = k) P(X_{i'} = k) = 365 \frac{1}{365} \frac{1}{365} = \frac{1}{365}$

$\Rightarrow P(X_i \neq X_{i'}) = 1 - P(X_i = X_{i'}) = 364/365$

3) $P(X_1 \neq X_N \cap X_2 \neq X_N \cap \cdots \cap X_{N-1} \neq X_N \mid X_1 \neq X_2 \neq \cdots \neq X_{N-1})\, P(X_1 \neq X_2 \neq \cdots \neq X_{N-1})$

$= P(X_1 \neq X_2 \neq \cdots \neq X_N)$ ②

4) Pas indepedants, en effet

$$P(X_1 \neq X_2 \neq X_3) = P(X_1 \neq X_2 \cap X_1 \neq X_3 \cap X_2 \neq X_3)$$

Q3

$$= \underbrace{P(X_1 \neq X_2)}_{= 1 - P(X_1 = X_2) = \frac{364}{365}} \underbrace{P(X_1 \neq X_3 \cap X_2 \neq X_3 \mid X_1 \neq X_2)}_{\frac{363}{365}}$$

$\neq P(X_1 \neq X_2) P(X_1 \neq X_3) P(X_2 \neq X_3) = \left(\frac{364}{365}\right)^2$

$\Rightarrow P(\underbrace{\text{"2 personnes au moins ont la même date d'anniversaire"}}_{A}) = 1 - P(X_1 \neq X_2 \neq X_3)$

$= 1 - \dfrac{364 \times 363}{(365)^2}$

5)

Si $N = 4$, $P(A) = 1 - P(X_1 \neq X_2 \neq X_3 \neq X_4)$

Q3

$$= 1 - \underbrace{P(X_4 \neq X_1 \cap X_4 \neq X_2 \cap X_4 \neq X_3 \mid X_1 \neq X_2 \neq X_3)}_{\frac{362}{365}} \times \underbrace{P(X_1 \neq X_2 \neq X_3)}_{\frac{363 \times 364}{(365)^2}}$$

Donc $\forall N$

$$P(A) = 1 - \frac{364}{365}\,\frac{363}{365} \cdots\cdots \frac{365-(N+1)}{365}$$

Exercice 5 :

1) $\left(\dfrac{1}{1-x}\right)' = \dfrac{1}{(1-x)^2}$; $\left(\dfrac{1}{1-x}\right)^{(2)} = \left(\dfrac{1}{(1-x)^2}\right)' = \dfrac{2(1-x)}{(1-x)^4} = \dfrac{2}{(1-x)^3}$

$\left(\dfrac{1}{1-x}\right)^{(3)} = \left(\dfrac{2}{(1-x)^3}\right)' = \dfrac{2 \times 3 \times (1-x)^2}{(1-x)^6} = \dfrac{2 \times 3}{(1-x)^4} \Rightarrow \left(\dfrac{1}{(1-x)}\right)^{(r)} = \dfrac{m!}{(1-x)^{m+1}}$

Puisque $\displaystyle\sum_{m=0}^{+\infty} x^m = \frac{1}{1-x} \Rightarrow \sum_{m=r}^{+\infty} m(m-1)\ldots(m-r+1)\, x^{m-r} = \frac{r!}{(1-x)^{r+1}}$

2)

$X_i = \{0,1\}$ $P(X_i = 1) = P(\text{Ivre} \cap \text{Contrôlé})$
$= P(\text{Ivre}) P(\text{contrôlé})$

$X_i \sim \text{Ber}\left(\frac{1}{20}\right) = \frac{1}{2} \times \frac{1}{10}$ $d x^r$

3)

S_m represente le nb de pt perdu après m jours. Par construction,

$S_m \sim \text{Bin}\left(m, \frac{1}{20}\right) \Rightarrow \mathbb{E}(S_m) = \frac{m}{20}$ et $\text{Var}(S_m) = \frac{m}{20}\,\frac{19}{20} = \frac{19m}{400}$

4) $T = \{6, 7, 8, \ldots\}$

$T = m$ si et seulement si $X_m = 1$ et $S_{m-1} = 5$

$\Rightarrow P(T = m) = P(X_m = 1 \cap S_{m-1} = 5) = P(X_m = 1)\, P(S_{m-1} = 5) = \dfrac{1}{20}\,\dfrac{(m-1)!}{5!(m-6)!}\left(\dfrac{1}{20}\right)^5\left(1 - \dfrac{1}{20}\right)^{m-6}$

indépendant

$T \in L^1$ Critère de d'Alembert : $\lim\limits_{m \to +\infty} \dfrac{u_{m+1}}{u_m} = \dfrac{19}{20} < 1$

et $\mathbb{E}(T) = \displaystyle\sum_{m=6}^{+\infty} \frac{m(m-1)!}{20\, 5!(m-6)!}\left(\frac{1}{20}\right)^5\left(\frac{19}{20}\right)^{m-6} = \frac{1}{20^6 \cdot 5!}\sum_{m=6}^{+\infty} m(m-1)\cdots(m-5)\left(\frac{19}{20}\right)^{m-6}$

$= \dfrac{1}{20^6 \cdot 5!}\,\dfrac{6!}{\left(1 - \frac{19}{20}\right)^7} = \dfrac{6}{20^6}\, 20^7 = 120$ (4 mois)

Exercice 6 : $\theta = \frac{100}{80} = \frac{5}{4}$ et $P(\text{"au moins 2 personnes } \geq 1,9\text{ m"}) = \alpha$

$$\Rightarrow \alpha = 1 - \frac{\theta^0}{0!}e^{-\theta} - \frac{\theta}{1!}e^{-\theta} = 1 - \left(1 + \frac{5}{4}\right)e^{-5/4} = 1 - \underbrace{P(\text{"0 personne} > 1,9\text{ m"})}_{P(X=0)}$$

$$= 0,355 \qquad\qquad = - \underbrace{P(\text{"1 personne} > 1,9\text{ m"})}_{P(X=1)}$$

Exercice 7 :

1) $P(X \leq n) = P(X=1) + P(X=2) + \dots + P(X=n))$

$$= \sum_{k=1}^{n} \frac{\alpha}{k(k+1)(k+2)} \Rightarrow P(X=n) = \frac{\alpha}{n(n+1)(n+2)}$$

2) $\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} \rightarrow A = \frac{1}{2} \quad B = -1 \quad C = \frac{1}{2}$

Donc $$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k+1} + \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k+2}$$

$$= \frac{1}{2}\left(\cancel{1} + \frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \dots + \cancel{\frac{1}{n}}\right)$$

$$- \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} - \dots - \cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$+ \frac{1}{2}\left(\cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \dots + \cancel{\frac{1}{n}} + \frac{1}{n+1} + \frac{1}{n+2}\right)$$

$$= \frac{1}{4} - \frac{1}{n+1} + \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2}\right) \Rightarrow \ell = \frac{1}{4}$$

3) Il faut que $\sum_{n=1}^{+\infty} \frac{\alpha}{n(n+1)(n+2)} = 1$

or ici $$\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \lim_{n \to +\infty} \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} = \frac{1}{4}$$

$$\Rightarrow \alpha = 4$$

4) $$|S_{2n} - S_n| = \left| \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \right| = \underbrace{\sum_{k=n+1}^{2n} \frac{1}{k}}_{n \text{ termes}} \geq n \frac{1}{2n} = \frac{1}{2}$$

En effet $\frac{1}{n+i} \geq \frac{1}{2n}$
$\Leftrightarrow 2n \geq n+i$
$\Leftrightarrow n \geq i$ (hypothèse)

$\Rightarrow \sum_{k=1}^{+\infty} \frac{1}{k}$ DV

5) $$E(X) = \sum_{n=1}^{+\infty} 4 \frac{\cancel{n}}{\cancel{n}(n+1)(n+2)} \underset{+\infty}{\sim} \sum^{+\infty} \frac{1}{n^2} \text{ CV}$$

$$= 4\sum_{n=1}^{+\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = 4\left(\frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \dots - \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} - \dots\right) = 2$$

6) $Var(X) = E(X^2) - E^2(X)$

or $$E(X^2) = 4\sum_{n=1}^{+\infty} \frac{n^2}{n(n+1)(n+2)} \underset{+\infty}{\sim} \frac{1}{n} \text{ DV d'après Q4}$$